Information and Computation Theory for Interdisciplinerss

This note compiles two well-known textbooks: Michael Sipser, *Introduction to the Theory of Computation (3e)* and Thomas M. Cover & Joy A. Thomas, *Elements of Information Theory (2e)*. No preknowledge is required.

Contents

1 Entropy

1.1

entropy $H(X)=-\sum\limits_{x\in\mathcal{X}}p(x)logp(x)=-Elogp(X)$

joint entropy, conditional entropy $H(X,Y)=H(X)+H(Y|X)$ *note:* usually $H(X|Y)\neq H(X|Y=y)$

relative entropy(KL distance) $D(p\|q)=-\sum\limits_{x\in\mathcal{X}}p(x)log\frac{p(x)}{q(x)}=-Elog\frac{p(X)}{q(X)}$ usually $D(p\|q)\neq D(q\|p)$

mutual inforamtion $I(X;Y)=D(p(x,y)\|p(x)p(y))$

$I(X;Y)=H(X)+H(Y)-H(X,Y)=H(X)-H(X|Y)$ $I(X;X)=H(X)$ Venn diagram

chain rule

$H(X_1,\cdots,X_n)=\sum\limits_{i=1}^{n}H(X_i|X_{i-1},\cdots,X_1),\ I(X_1,\cdots,X_n;Y)=\sum\limits_{i=1}^{n}I(X_i;Y|X_{i-1},\cdots,X_1)$

Jesen Ineq: $\text{convex func. } f \text{ and a RV } X,\ Ef(X)\geq f(EX)$

Info. Ineq: $D(p\|q)\geq 0,\ \text{eq. iff } p=q\ \ I(X;Y)\geq 0,\ \text{eq. iff } X\ id.\ Y$

$H(X)\leq log|\mathcal{X}|\ \ H(X_1,\cdots,X_n)\leq\sum\limits_{i=1}^{n}H(X_i)$

log-sum Ineq: $\sum_{i=1}^{n}a_i\log\frac{a_i}{b_i}\geq\left(\sum_{i=1}^{n}a_i\right)\log\frac{\sum_{i=1}^{n}a_i}{\sum_{i=1}^{n}b_i}$ $D(p\|q)$ is convex about (p,q) and $H(p)$ is concave about p.

Markov chain $X\rightarrow Y\rightarrow Z,\ I(X;Z|Y)=0$ Data Processing Ineq: $I(X;Y)\geq I(X;Z),\ \text{eq. iff } I(X;Y|Z)=0$

$\theta\rightarrow X\rightarrow T(X)$ sufficient statistics $I(\theta;X)=I(\theta;T(X))$ or $X\ id.\ \theta \text{ when } T(X)$ is given
minimal suf. stat. $\theta\rightarrow T(X)\rightarrow U(X)\rightarrow X$

Fano Ineq:
$$X\rightarrow Y\rightarrow\hat{X},\ P_e=Pr(X\neq\hat{X}),\ H_e=-P_elogP_e-(1-P_e)log(1-P_e),$$
$$H(X|Y)\leq H(X|\hat{X})\leq H_e+P_eH(X)\leq H_e+P_elog|\mathcal{X}|$$

$X \sim p(x),\ Y \sim q(y),\ Pr(X=Y) \geq 2^{-H(p)-D(p\|q)}$

1.2

(weak)LLN, asymptotic equipartition property AEP:
$X_1,\cdots,X_n \overset{iid.}{\sim} p(x),\ \ -\frac{1}{n}logp(X_1,\cdots,X_n) \overset{p}{\to} H(X)$

typical set $A_\epsilon^n \subset \mathcal{X}^n$ $\text{ large } n,\ Pr(A_\epsilon^n) > 1-\epsilon,\ (1-\epsilon)2^{n(H(X)-\epsilon)} \leq |A_\epsilon^n| \leq 2^{n(H(X)+\epsilon)}$ (In short, A_ϵ^n has roughly 2^{nH} elements with equal prob. 2^{-nH}.)

Nearly nH bits can express sequence X^n.(typical set $n(H+\epsilon)+1$ bit and nontypical $nlog|\mathcal{X}|+1$ bit)

In the sense of first-order exponent, among all sets that have $Pr > 1-\epsilon,\ \ A_\epsilon^n$ is the minimal.

stationary stochastic process, stationary Markov chain

entropy rate of a sto. $\{X_i\}$: $H(\mathcal{X}) = \lim\limits_{n\to\infty} \frac{1}{n}H(X_1,\cdots,X_n)$(if limit exists)

for stn. sto. also $= \lim\limits_{n\to\infty} H(X_n|X_{n-1},\cdots,X_1)$ *proof: Stolz-Cesaro Means Th.*

for stn. MC of trans. mx P and stn. dis. $\mu = P\mu$, $H(\mathcal{X}) = H(X_2|X_1) = -\sum\limits_{ij} \mu_i P_{ij} log P_{ij}$

Second law of thermodynamics: as time $n\uparrow$,

$$D(\mu_n\|\mu_n')\downarrow,\ \text{esp. } D(\mu_n\|\mu);$$
$$\text{if stn. dis. is uni.(equal a priori prob. principle) } H(X_n)\uparrow;$$
$$\text{for stn. MC } H(X_n|X_1)\uparrow;$$
$$\text{operator(e.g. shuffling) } T\ id.\ \ X,\ \ H(TX)\geq H(X)$$

$$\text{stn. MC } \{X_n\},\ Y_i=\phi(X_i),\ H(\mathcal{Y}) = \lim_{n\to\infty} H(Y_n|Y_{n-1},\cdots,Y_1) = \lim_{n\to\infty} H(Y_n|Y_{n-1},\cdots,Y_1,X_1)$$

Shannon-McMillan-Breiman(general AEP) Th: H is the entropy rate of a finite ergodic sto. X_n, $-\frac{1}{n}logp(X_0,\cdots,X_{n-1}) \overset{a.s.}{\to} H$ *proof: Sandwich Th.*

1.3

differential entropy $h(X) = -\int_S f(x)logf(x)dx$ (S is the support set of RV X)

joint diff. ent., conditional diff. ent., relative ent., MI, AEP are in the same way.

example: $h(\mathcal{N}(\mu,\Sigma)) = \frac{n}{2} + \frac{1}{2}log(2\pi)^n|\Sigma|$; for bivar. $\mathcal{N}_2$, $I(X_1;X_2) = -\frac{1}{2}log(1-\rho^2)$

$h(aX) = h(X) + log|a|,\ h(AX) = h(X) + log|detA|$

Among all dis. with $\Sigma = EXX'$, Gauss dis. has max ent. $h(X) \leq \frac{1}{2}log(2\pi e)^n|\Sigma|$ *proof:* $D(f\|N)\geq 0,\ \ \int flogN = \int NlogN$

estimation error (if have side info. Y) $E(X-\hat{X})^2 \geq \frac{1}{2\pi e}e^{2h(X|Y)}$

Similarly, $A_\epsilon^n = \{x\in S^n : -\frac{1}{n}logp(x) - h(X)| \leq \epsilon\}$. While in discrete case we use cardinal $|A_\epsilon^n| \doteq 2^{nH}$, in continuous case we use volume $V(A_\epsilon^n) \doteq 2^{nh}$ (like a cube with side length 2^h).

relation with discrete ent: dis. $p(x)$ is Riemann integrable, X is partitioned as X^Δ by intervals with length Δ, then $\lim\limits_{\Delta\to 0} H(X^\Delta) + log\Delta = h(X)$. Thus a n bit quantized cont. RV X has a ent. of $h(X)+n$.

more: several ineq. about det can be derived thru ent. of a multivar ndis. *e.g.* Hadamard ineq: $\prod\limits_i \Sigma_{ii} \geq |\Sigma| \geq \prod\limits_i \sigma_i^2$ where σ_i^2 is the cond. variance of X_i given other X_j, or $\sigma_n^2 = \frac{|\Sigma_n|}{|\Sigma_{n-1}|}$

1.4

Maximun entropy dis./prin./estimation

$\max_f h(f)$ s.t. $f(x) \geq 0,\ \int_S f(x)dx = 1,\ \int_S f(x)r_i(x)dx = \alpha_i$

$f^*(x) = e^{\lambda_0 + \sum_i \lambda_i r_i(x)} = \frac{e^{\sum_i \lambda r_i(x)}}{\int_S e^{\sum_i \lambda r_i(x)} dx}$ *proof: Lagrange; Info ineq.*

example: Boltzmann dis. $S = [0, +\infty),\ EX = \mu,\ f(x) = \frac{1}{\mu} e^{-\frac{x}{\mu}}$

$S = (-\infty, +\infty),\ EX = \alpha_1,\ EX^2 = \alpha_2,\ f(x) = \mathcal{N}(\alpha_1, \alpha_2 - \alpha_1^2)$ but when we have third moment constraint, λ_3 needs to be 0 to aviod $\int_{-\infty}^{+\infty} f = \infty$, and therefore this method may be out of work. However we can add some carefully devised perturbation onto the original $\mathcal{N}$ to get whatever α_3 while holding α_1 and α_2. Thus $\sup h(f) = h(\mathcal{N}(\alpha_1, \alpha_2 - \alpha_1^2)) = \frac{1}{2} \ln 2\pi e(\alpha_2 - \alpha_1^2)$, the max ent. is only ϵ-reachable.

diff. ent. rate $h(\mathcal{X}) = \lim_{n\to\infty} \frac{1}{n} h(X_1, \cdots, X_n) \overset{stn.}{=} \lim_{n\to\infty} h(X_n|X^{n-1})$

for Gauss stno. $h(\mathcal{X}) = \frac{1}{2} log 2\pi e + \frac{1}{4\pi} \int_{-\pi}^{\pi} log S(\lambda) d\lambda$, $\sigma_\infty^2 = \frac{1}{2\pi e} 2^{2h}$(best em. error given inf. history)

AR model, autocorrelation func. $R(k) = EX_i X_{i+k}$ power spectral density PSD $S(\lambda) = \mathcal{F}(R(k))$

Burg Th:
$$\begin{gathered} p \text{ order Gaussian-Markov autoreg. sto.} \\ X_i = \sum_{j=1}^{p} a_j X_{i-j} + Z_i,\ Z_i \overset{iid.}{\sim} \mathcal{N}(0, \sigma^2) \\ \text{attains the max ent. rate among all sto. satisfying the conditions} \\ EX_i X_{i+k} = \alpha_k, 1 \leq k \leq p, \forall i \end{gathered}$$

a_i, σ^2 can be solved from Yule-Walker equations: $R(m) = \sum_{k=1}^{p} a_k R(m-k) + \sigma^2 \delta_{m,0}, 1 \leq m \leq p$

2 Coding, Statistics and Investment

2.1

source coding $C : \mathcal{X} \to \mathcal{D}^*,\ l(x) = |C(x)|,\ L(C) = El(X)$ (alphabet $\mathcal{D} = \{1, \ldots, D-1\}$)

nonsigular $\supset$ uniquely decodable $\supset$ instantaneous/prefix

Kraft ineq: for inst. code on D, code word length $l_1, \ldots$, $\sum_i D^{-l_i} \leq 1$ *proof: all code words' son sets disjoint.*

optimal code $H_D(X) \leq L < H_D(X) + 1,$ eq. iff. $D^{-l_i} = p_i$ D-adic dis. *proof: Shannon coding by Lagrange*

Shannon coding: $l_i = \lceil log_D \frac{1}{p_i} \rceil$ (to prove the upper bound)

for stno. $\{X_n\}, L \to H(\mathcal{X})$ Shannon First Th (Noiseless Coding Th): $H(\mathcal{X}) \leq L \leq H(\mathcal{X}) + \frac{1}{n}$

for wrong code(using $q(x)$), the length will increased by $D(p\|q)$

Acc. all uni. decodable codes satisfy Kraft ineq.(McMillian ineq.) so they are no better than prefix codes.

Huffman coding C_H is optimal. Shannon-Fano-Elias coding

Shannon coding is competitive optimal $Pr(l(X) \geq l'(X) + c) \leq \frac{1}{2^{c-1}}$

(refer to *5.5* for Kolmogorov complexity)

2.2

prob. simplex $\mathcal{P}$ type P_x on $\mathcal{X}$, type class $T(P) = \{x \in \mathcal{X}^n : P_x = P\},\ P \in \mathcal{P}^n$

1. $|\mathcal{P}^n| \le (n+1)^{|\mathcal{X}|}$ 2. $Q^n(x) = 2^{-n(D(P_x\|Q)+H(P_x))}$ 3. $|T(P)| \doteq 2^{nH(P)}$ 4. $Q^n(T(P)) \doteq 2^{-nD(P\|Q)}$

$(x = \{x_1, \dots, x_n\},\ X_i \overset{iid.}{\sim} Q(x))$

seq. typical set $T_\epsilon^{Q^n} = \{x : D(P_x\|Q) \le \epsilon\},\ Pr(T_\epsilon^{Q^n}) \to 1,\ D(P_x\|Q) \overset{a.s.}{\to} 0$

Large deviation theory Sanov Th:

$$\text{subset } E \subset \mathcal{P},\ Q^n(E) \le (n+1)^{|\mathcal{X}|} 2^{-nD^*},\ D^* = \min_{P \in E} D(P\|Q);$$

$$\text{if } E \text{ is the closure of its interior},\ -\frac{1}{n} logQ^n(E) \to D^*$$

if constraints $E = \{P : \sum_x P(x) g_i(x) \ge \alpha_i\}$, we can get $P^*(x) = \frac{Q(x) e^{\sum_i \lambda g_i(x)}}{\sum_{x \in \mathcal{X}} Q(x) e^{\sum_i \lambda g_i(x)}}$ using Lagrange.

jointly typical set

$$A_\epsilon^n = \{(x^n, y^n) \in \mathcal{X}^n \times \mathcal{Y}^n : \left|-\frac{1}{n}\log p(x^n) - H(X)\right| < \epsilon,\ \left|-\frac{1}{n}\log p(y^n) - H(Y)\right| < \epsilon,$$

$$\left|-\frac{1}{n}\log p(x^n, y^n) - H(X,Y)\right| < \epsilon\},\ Pr(A_\epsilon^n) \to 1$$

$(\tilde{X}^n, \tilde{Y}^n) \sim p(x^n)p(y^n),\ Pr((\tilde{X}^n, \tilde{Y}^n) \in A_\epsilon^n) \to 2^{-nI(X;Y)}$ *proof: Sanov Th.*

closed convex set $E \subset \mathcal{P},\ Q \notin E,\ P^* = \arg\min_{P \in E} D(P\|Q),\ \forall P \in E,\ D(P\|Q) \ge D(P\|P^*) + D(P^*\|Q)$

Conditional Limit Th: seq. $X^n \overset{iid.}{\sim} Q,\ n \to \infty,\ Pr(X_i = x | P_{X^n} \in E) \overset{p}{\to} P^*(x)$ (i.e. type P^* represents the whole set) *proof:* $D(P_1\|P_2) \ge \frac{1}{2\ln 2}\|P_1 - P_2\|_1^2$, *D's convergence implies $\mathcal{L}_1$ norm's convergence. (Taylor: $D(P_1\|P_2) = \frac{1}{2}\chi^2_{P_1,P_2} + \dots$)*

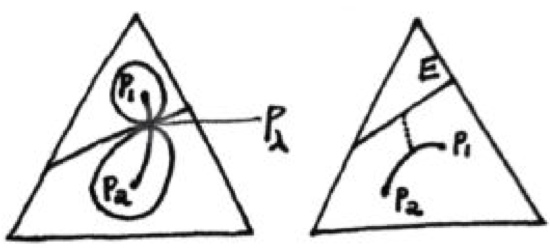
Hypo test $X_i \overset{iid.}{\sim} Q(x),\ H_1 : Q = P_1,\ H_2 : Q = P_2$, Neyman-Pearson Lem: likelihood ratio $T \ge 0$,

acceptance region $A_n(T) = \{x^n : \frac{P_1(x^n)}{P_2(x^n)} > T\},\ \alpha^* = P_1^n(A_n^c(T)),\ \beta^* = P_2^n(A_n(T)),$

$$\text{other regions with } \alpha \le \alpha^* \text{ must have } \beta \ge \beta^*$$

$\frac{P_1(X^n)}{P_2(X^n)} > T$ equals to $D(P_{X^n}\|P_2) - D(P_{X^n}\|P_1) > \frac{1}{n}logT$, under this constraint we minimize $D(P\|P_2)$(also $D(P\|P_1)$) using Lagrange and get $P_\lambda = \frac{P_1^\lambda(x)P_2^{1-\lambda}(x)}{\sum_{x \in \mathcal{X}} P_1^\lambda(x)P_2^{1-\lambda}(x)}$ to estimate

$\alpha_n \doteq 2^{-nD(P_\lambda\|P_1)}$, $\beta_n \doteq 2^{-nD(P_\lambda\|P_2)}$. ($\lambda$ can be determined by $D(P_{X^n}\|P_2) - D(P_{X^n\|P_1}) = \frac{1}{n}logT$.)



relative ent. AEP: $X_1, \cdots, X_n \overset{iid.}{\sim} P_1(x),\ \forall P_2(x),\ -\frac{1}{n}log\frac{P_1(X_1,\cdots,X_n)}{P_2(X_1,\cdots,X_n)} \overset{p}{\to} D(P_1\|P_2)$

relative ent. typical set $P_1(A_\epsilon^n(P_1\|P_2)) > 1 - \epsilon,\ P_2(A_\epsilon^n(P_1\|P_2)) \to 2^{-nD(P_1\|P_2)}$

Chernoff-Stein Lem:

$$\alpha_n = P_1^n(A_n^c),\ \beta_n = P_2^n(A_n),\ \beta_n^\epsilon = \min_{A_n \subset \mathcal{X}, \alpha_n < \epsilon} \beta_n,\ \lim_{n\to\infty} \frac{1}{n} log\beta_n^\epsilon = -D(P_1\|P_2)$$

when Bayesian weighted $D^* = \min_{A_n} \lim_{n \to \infty} -\frac{1}{n}log(\pi_1\alpha_n + \pi_2\beta_n)$, Chernoff Info.

$C(P_1, P_2) = D^* = D(P_\lambda\|P_1) = D(P_\lambda\|P_2)$ or $C(P_1, P_2) = -\min_{0 \le \lambda \le 1} log(\sum_x P_1^\lambda(x) P_2^{1-\lambda}(x))$

(*note:* D^* is not related to π_1, π_2 since large sample will eliminate a priori knowledge $\frac{\pi_1}{\pi_2}\frac{P_1(X_n)}{P_2(X_n)} \overset{?}{\sim} T$)

Score func. $V=\frac{\partial}{\partial\theta}\ln f(X;\theta),\ EV=0$ Fisher Info. $J(\theta)=EV^2=-E\frac{\partial^2}{\partial\theta^2}\ln f(X;\theta)$
$J_n(\theta)=nJ(\theta)$

Cramer-Rao Ineq: $\text{unbiased stat. } T(X) \text{ of } \theta,\ \Sigma(T)\geq J^{-1}(\theta)$ (for multivar. it means mx $\Sigma-J^{-1}$ is semipositive.)(Similarly, for biased stat. we have $b_T(\theta)=ET-\theta,\ E(T-\theta)^2\geq\frac{(1+b'_T(\theta))^2}{J(\theta)}+b_T^2(\theta)$.)

proof: Cauchy-Schwarz Ineq. for $V-EV$ and $T-ET$.

some senses: for para. dis. family $\{p_\theta(x)\}$, $\theta\to\theta',\ D(p_\theta\|p'_\theta)\sim\frac{J(\theta)}{2}$; de Brujin Ineq.
$Z\ id.\ X,\ Z\sim\mathcal{N}(0,1),\ \frac{\partial}{\partial t}h(X+\sqrt{t}Z)=\frac{1}{2}J(X+\sqrt{t}Z),\ \text{if limit exists, } \frac{\partial}{\partial t}h(X+\sqrt{t}Z)\big|_{t=0}=\frac{1}{2}J(X)$ (h's base is e); Just like ent. power $2^{nH(X)}$ can be seemed as the volume of typical set, Fisher info. $J(X)$ can be seemed as the surface area, where $J(X)=\int\frac{(\frac{\partial f}{\partial x})^2}{f}dx$; Fisher info's convolution ineq. $\frac{1}{J(X+Y)}\geq\frac{1}{J(X)}+\frac{1}{J(Y)}$

ent. powe,r Ineq: $X\ id.\ Y,\ \dim X=\dim Y=n,\ 2^{\frac{2}{n}h(X+Y)}\geq 2^{\frac{2}{n}h(X)}+2^{\frac{2}{n}h(Y)}$ or $h(X+Y)\geq h(X'+Y')$, where $X',Y'\sim\mathcal{N},\ X'\ id.\ Y',\ h(X')=h(X),\ h(Y')=h(Y)$

2.3

Kelly game $b^*=p$ Gambling conservation Th: $W^*+H=logm$ (for uniform fair oppo. game)

estimation of entropy of English (Shannon letter guessing game)

potfolio $\mathcal{B}=\{b\in\mathcal{R}^m:b_i\geq 0,\sum_{i=1}^m b_i=1\}\ X\sim F(x),\ S=b'X$

first and second moment method: Sharpe-Markowitz theory, CAPM

growth rate $W(b,F)=\int logSdF=Elogb'X\ \ S_n=\prod_{i=1}^n S_i,\ \frac{1}{n}logS_n\overset{a.s.}{\to}W,\ S_n\doteq 2^{nW}$

log optimal porfolio $W^*(F)=\max_b W(b,F)$

$W(b,F)$ is concave about b, linear about F, and $W^*(F)$ is convex about F.

b^*'s KT condition: $E(\frac{b'X}{b^{*\prime}X})\leq 1,\ E(\frac{X_i}{b^{*\prime}X})=1 \text{ if } b_i^*>0,\leq 1 \text{ if } b_i^*=0$

causal portfolio $b_i:\mathcal{R}_+^{m(i-1)}\to\mathcal{B}$ log optimal is the best. $ElogS_n^*=nW^*\geq ElogS_n$

Side info. raises growth rate. $\Delta W=\int_y f(y)\Delta W_{Y=y}\leq I(X;Y)$

$W_\infty^*=\lim_{n\to\infty}\frac{1}{n}W^*(X_1,\cdots,X_n)\overset{stn.}{=}\lim_{n\to\infty}W^*(X_n|X^{n-1})$

$\frac{S_n}{S_n^*}\text{ is a supermartingale, }\overset{a.s.}{\to}V,\ EV\leq 1,\ Pr(\sup_n\frac{S_n}{S_n^*}\geq t)\leq\frac{1}{t}$

universal portfolio:
$$S_n^*(x^n)=\max_b\prod_{i=1}^n b'x_i,\ \hat{S}_n(x^n)=\prod_{i=1}^n\hat{b}_i'(x^{i-1})x_i,\ \max_{\hat{b}}\min_{x^n}\frac{S_n(x^n)}{S_n^*(x^n)}=V_n,$$
$$V_n=(\sum_{n_1+\cdots+n_m=n}\binom{n}{n_1,\ldots,n_m}2^{-nH(\frac{n_1}{n},\ldots,\frac{n_m}{n})})^{-1}\sim n^{-\frac{m-1}{2}}$$

$\hat{b}_{n+1}(x^i)=\frac{\int_\mathcal{B}bS_i(b,x^i)d\mu(b)}{\int_\mathcal{B}S_i(b,x^i)d\mu(b)},\ \hat{S}_n(x^n)=\int_\mathcal{B}S_n(b,x^n)d\mu(b)$

3 Communication

3.1

discrete channel $(\mathcal{X},p(y|x),\mathcal{Y})$ DMC n-th extension $p(y_k|x^k,y^{k-1})=p(y_k|x_k)$, non-feedback $p(x_k|x^{k-1},y^{k-1})=p(x_k|x^{k-1})$, thus $p(y^n|x^n)=\prod_i p(y_i|x_i)$

(M,n) code of channel: message index set $W \in \mathcal{W} = \{1,\dots,M\}$, coding func. $X^n: \mathcal{W} \to \mathcal{X}^n$ and codebook $\mathcal{C} = \{x^n(1),\dots,x^n(M)\}$, decoding func. $g: \mathcal{Y} \to \mathcal{W}$

$W \to X^n(W) \to Y^n \to \hat{W}$

conditional, maximum, average prob. of error
$\lambda_i = \sum_{y^n} p(y^n|x^n(i)) I(g(y^n) \neq i),\ \lambda = \max_i \lambda_i,\ P_e^n = \bar{\lambda_i}$

rate $R = \frac{logM}{n}$ bit/trans. achievable rate $n \to \infty,\ \lambda \to 0$

channel capacity $C = \max_{p(x)} I(X;Y)\ \ 0 \le C = \max_{p(x)} H(Y) - H(Y|X) \le \min(log|\mathcal{X}|, log|\mathcal{Y}|)$

note: we use max not sup here since $I(X;Y)$ is a concave func. on convex set of $p(x)$ and several algo. can compute this maximum.

direct understanding: For each typical seq. X^n there are roughly $2^{nH(Y|X)}$ seq. of Y^n corresponding to it, and the total number of Y^n (typical) is $2^{nH(Y)}$, so nearly $2^{nI(X;Y)}$ disjointed image sets of diff. inputs X^n can be seperated in one transmission. (or think of jointly typical $Pr = 2^{-nI}$)

example: BSC $C = 1 - H(p)$, BEC $C = 1 - \alpha$ symmetric channel

Channel Coding Th (Shannon Second Th):
$\text{DMC}, \forall R < C,\ \exists (2^{nR}, n) \text{ code}, \lambda \to 0;\ \text{Conversely}, \forall (2^{nR}, n) \text{ code with } \lambda \to 0 \text{ must has } R \le C$

proof: randomly generated codebook, jointly typical decoding, so error comes from either not jointly typical Y^n *or other possible inputs that are jointly typical with:* $Pr(V^n \neq \hat{V}^n) = Pr((X^n(i), Y^n) \notin A_\epsilon^n) + \sum_{j \neq i} Pr((X^n(j), Y^n) \in A_\epsilon^n)$*; for converse th, Fano ineq. and Data-processing ineq. lead to*
$nR = H(W^{uni.}) \le 1 + P_e^n nR + nC$. (strong converse edition:
$R < C,\ P_e^n \to 0;\ R > C,\ P_e^n \to 1$)

note: equality needs 1. coding $X^n(W)$ and decoding $\hat{W}$ are sufficient(all diff.); 2. $Y_i\ id.$; 3. X_i's dis. is $p^*(x)$.

Hamming code, error-detecting code minimum weight and minimun distance parity check mx.
$H(c + e_i) = He_i$ systematic code (n,k,d)
$\text{e.g. Hamming } r(H) = l,\ n = 2^l - 1,\ k = 2^l - l - 1,\ d = 3$ block code and convolutional code
more: BCH code, LDPC code, turbo code.

feedback code $x_i(W, Y^{i-1})$, feedback capacity $C_{FB} = C$ (feedback can simplify coding but cannot enlarge capacity of DMC.)

Source-Channel Seperation Th: ($Pr(V^n \neq \hat{V}^n) = \sum_{v^n} p(v^n) \lambda_{v^n}$)

$$\{V^n\} \text{ satisfies AEP (ergodic stno.)},\ H(\mathcal{V}) < C,\ \exists \text{ source-channel code},\ Pr(V^n \neq \hat{V}^n) \to 0,\ \text{vice versa}$$

Thus two-step way is equally efficient. First we do data compressing (from AEP): nearly all prob. is in a seq. set with size 2^{nH}, and we can use $R > H$ code to express this info. source with little error. Second we do data transmitting (from Joint AEP): for large grouping length n, nearly all inputs and outputs are jointly typical with 2^{-nI} prob. of exception, and we can use $R < \max I = C$ code to keep error prob. low. This Th. $H < C$ combines the two, telling that we can devise source code(exprssing efficiently) and channel code(confronting noise) seperatedly.

3.2

Gaussian channel $Y_i = X_i + Z_i,\ Z_i \sim \mathcal{N}(0, N)$ power constraint $\frac{1}{n}\sum_i x_i^2 \le P$

$C = \max_{f(x): EX^2 \le P} I(X;Y) = \frac{1}{2} log(1 + \frac{P}{N})$ bit/trans.

proof:
$EY^2 = P + N,\ I(X;Y) = h(Y) - h(Z),\ \text{when } Y \sim \mathcal{N}(0, P+N) \text{ i.e. } X \sim \mathcal{N}(0,P) \max$

Similarly, code $(2^{nR}, n)$ with $R < C$ is achievable. (Each decoding ball's radius is $\sqrt{nN}$ and outputs' $\sqrt{n(P+N)}$, so the number of disjointed balls is no more than $(\frac{P+N}{N})^{\frac{n}{2}}$.)

finite bandwidth W Nyquist-Shannon Sampling Th: signal $f(t)$ with maximum cut-off freq. W can be completely determined by sampling seq. of $\frac{1}{2W}$ s time interval. Thus it can be seemed as a vec. in $2WT$ dof/dim space.

bandwidth W, noise psd $\frac{N_0}{2}$, noise power N_0W (spherical ndis. with covarmx $\frac{N_0}{2}I$) AWGN channel

$C = Wlog(1 + \frac{P}{N_0W})$ bit/s (Shannon Formula) $W \to \infty,\ C = W \cdot SNR = \frac{P}{N_0}$ nat/s

parallel Gaussian channel $\sum EX^2 \le P,\ C = \max I(X^k; Y^k)$ max power allocation:
$P_i = (v - N_i)^+,\ \sum(v - N_i)^+ = P$ (water-filling)

correlated noise (memory channel can also convert to this)
$\frac{1}{n}tr(\Sigma_X) \le P,\ C_n = \max \frac{1}{2n}log\frac{|\Sigma_X + \Sigma_Z|}{|\Sigma_Z|}$, also sloved by water filling onto Σ_Z's eigen values λ_i.
$C_n = \frac{1}{2n}\sum\limits_{i=1}^{n} log(1 + \frac{(\lambda - \lambda_i)^+}{\lambda_i}),\ \sum\limits_{i=1}^{n}(\lambda - \lambda_i)^+ = nP$

more: for stno, covarmx is Toeplitz mx, when $n \to \infty$ the envelop of its eigenvalues approaches the power spectral $N(f)$ of this stno. ; feedback Gaussian channel $C_{n,FB} = \max\limits_{tr(\Sigma_X) \le nP} \frac{1}{2n}log\frac{|\Sigma_{X+Z}|}{|\Sigma_Z|}$, X^n is no longer id. with Z^n and $X = BZ + V$ miximizes. $C_{n,FB} \le \min(C_n + \frac{1}{2}, 2C_n)$ is only slightly higher than C_n.

3.3

reproduction/code point $\hat{X}(X)$, Dirichlet partition Lloyd algo.

distortion measure $d(x, \hat{x})$ Hamming distortion, squared error distortion

$(2^{nR}, n)$ rate distortion code (R, D) achievable $\lim\limits_{n\to\infty} Ed(X^n, g_n(f_n(X^n))) \le D$

rator. func. $R(D) = \inf\limits_{D} \text{achi.} R = \min\limits_{p(\hat{x}|x):D} I(X;\hat{X})$ (Shannon Third Th. $R \ge R(D_0) \Leftrightarrow D \le D_0$)

example: (Ham. distortion, $R(D) = 0$ at other large D) $B(p)$ source:
$R(D) = H(p) - H(D),\ 0 \le D \le \min(p, 1-p)$; $\mathcal{N}(0, \sigma^2)$ source:
$R(D) = \frac{1}{2}log\frac{\sigma^2}{D},\ 0 \le D \le \sigma^2$ (similarly, ball of radius $\sqrt{nD}$ filling in ball of radius $\sqrt{n\sigma^2}$, the number of codewords equals to $2^{nR(D)}$); parallel(multindis.) source:
$R(D) = \sum\limits_i \frac{1}{2}log\frac{\sigma_i^2}{D_i},\ D_i = \min(\lambda, \sigma_i^2),\ \sum\limits_i D_i = D$ (i.e. anti-waterfilling on the spectral)

Similarly, for combined source and channel coding, $D = \frac{1}{n}\sum\limits_{i=1}^{n} Ed(V_i, \hat{V}_i)$ can be achieved iff

$C > R(D)$.

more: rator. is achi. when grouping length n is enough.(so put them together to describe will have less distortion than considering seperatedly) distortion typical set, strong typical set Blahut-Arimoto algo. for computing rator func.

universal source code: $\exists(2^{nR}, n) \text{ code}, \forall \text{ source } Q \text{ with } H(Q) < R,\ P_e^n \to 0$

more: minimax redundancy, Lemple-Ziv(LZ) coding

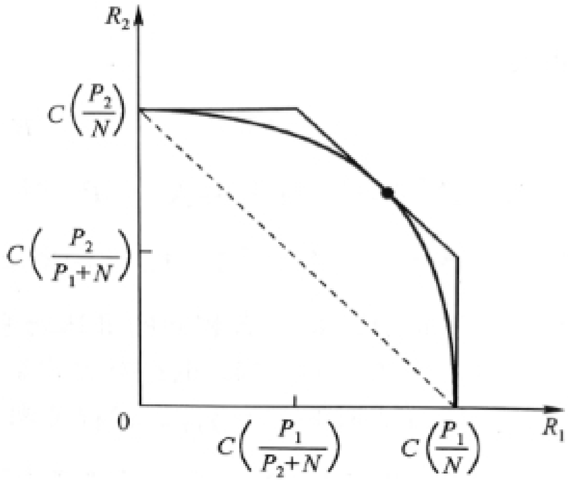
multiaccess channel, broadcast channel, relay channel, interference channel multiaccess: *example:* binary addition and multiplication channel capacity region $\boldsymbol{R} \in \mathcal{C}$ i.e. convex hull of

$R(S)=\sum_{i\in S} R_i,\ X(S)=\{X_i:\in S\},\ \forall S\subset\{1,\dots,m\},\ R(S)\le I(X(S);Y|X(S^c)) \Leftrightarrow P_e^n\to 0$

onion-peeling at corner points

for Gausssion, denote $C(x)=\frac{1}{2}log(1+x)$, $\sum_{i\in S} R_i\le C(\frac{\sum_{i\in S} P_i}{N})$; total code-rate $C(\frac{mP}{N})$ will approach infty when $m\to\infty$ but mean of each sender will approach 0. CDMA(the polyline), FDMA and TDMA(the curve)

for source coding, Slepian-Wolf Th: $\forall S\subset\{1,\dots,m\},\ R(S)>H(X(S)|X(S^c)) \Leftrightarrow P_e^n\to 0$



digest: Shannon's three theorems. 1. non-distortion/lossless length-variable source-coding: (unidecodable) $R>H$ (L is rate R) 2. noisy channel-coding: $R<C$ (AWGN $C=B\log(1+\frac{S}{N})$) 3. fidelity-criteria/lossy source-coding: $R>R(D)$

4 Automaton and Language

4.1

computation model finite automation FDA

transition function $\delta:Q\times\Sigma\to Q$, accept state, languge $L(M)=A$

regular language regular operation: union $\cup$, concatenation $\circ$, star *

nondeterministic ϵ NFA $\delta:Q\times\Sigma_\epsilon\to\mathcal{P}(Q)$

$NFA=DFA$

REG's closure under regular operation

regular expression REX: $R=a\in\Sigma \text{ or } \epsilon \text{ or } \varnothing \text{ or } R_1\cup R_2 \text{ or } R_1\circ R_2 \text{ or } R_1^*$

token, lexical analyzer

GNFA (like contraction) $\delta:(Q-\{q_{acc}\})\times(Q-\{q_{acc}\})\to\mathcal{R}$

equivalence: a language is regular iff it can be expressed by regex. *proof: construction in closure; GNFA*

irregular lang. *example:* $A=\{0^n1^n|n\ge 0\}$

Pumping Lem: $\begin{matrix}\text{lang.}A\in REG,\ \exists p,\ \forall\text{ string } s \text{ with a length of no less than } p, s=xyz \text{ and:}\\ 1.\forall i\ge 0,\ xy^iz\in A\quad 2.|y|>0\quad 3.|xy|\le p\end{matrix}$

4.2

parser, context-free grammar, context-free language CFL parse tree leftmost derivation ambiguous, inherently ambig.

Chomsky normal form: $A \to BC$ or $A \to a$ (or $S \to \epsilon$)

pushdown automaton PDA stack $\delta : Q \times \Sigma_\epsilon \times \Gamma_\epsilon \to \mathcal{P}(Q \times \Gamma_\epsilon)$

equivalence: a language is context-free iff it can be recognized by a PDA. *proof: sign symbol $\$$, nondeter. substitution and comparison; A_{pq} for string that brings PDA from state p and empty stack to q and empty stack $\to A_{pr}A_{rq}$ or $\to aA_{rs}b$*

$REG \subset CFL$

CFL's Pumping Lem: $\ldots s = uvxyz$ and: $1. \forall i \geq 0,\ uv^ixy^iz \in A \quad 2. |vy| > 0 \quad 3. |vxy| \leq p$

more: DPDA, DCFL leftmost reduction, valid string, handle, forced handle, DCFG

more: dotted rule DK-test almost(end sign lang.) equivalence of DCFG and DPDA LR(k) grammar

5 Computability

5.1

Turing machine, configuration $L(M)$

Turing-recognizable, decidable

variants and robustness: multitape TM, nondeterministic TM, enumerator recursive enumerable

algorithm, Hilbert's problem, Church-Turing Thesis

description of TM

5.2

decidable problem(language): $A,\ E,\ EQ$ for $DFA(REX),\ CFG$ except EQ_{CFG}

$REG \subset CFL \subset DECI \subset RE$

universal TM A_{TM} is undecidable. *proof: contradiction, Cantor diagonal method*

The set of all TM $\{\langle M \rangle\}$ is countable but the set of all lang. $\mathcal{L}$ is uncountable, thus $\exists\ lang.\ A \notin RE$.

$A,\ \overline{A} \in RE \Rightarrow A \in DECI \quad \overline{A_{TM}} \notin RE$

5.3

reduction undeci: $HALT_{TM},\ E_{TM},\ REG_{TM},\ EQ_{TM}$ *proof: reduct to A_{TM} etc.*

Rice Th: P is a non-trivial property, $L_p = \{\langle M \rangle | L(M) \in P\}$ is undeci. (not all TM descriptions belong to set P and $L(M_1) = L(M_2),\ \langle M_2 \rangle \in P$ iff $\langle M_1 \rangle \in P$.)

computation history LBA A_{LBA} is deci. but E_{LBA} is undeci.

more: $ALL_{CFG},\ PCP$

computable function mapping(many-one) reducibility:
$\exists$ computable func. $f : \Sigma^* \to \Sigma^*,\ \forall \omega,\ \omega \in A \Leftrightarrow f(\omega) \in B \ \ A \leq_m B$

If A is undeci/unre then B is undeci/unre; if B is deci/re then A is deci/re.

$A \leq_m B \Leftrightarrow \overline{A} \leq_m \overline{B}$ *example:* $A_{TM} \leq_m \overline{EQ_{TM}}$

5.4

$SELF$ machine that obtains its own description

Recursion Th:

$T \text{ is a TM of func. } t: \Sigma^* \times \Sigma^* \to \Sigma^*, \ \exists \text{ TM R of func. } r: \Sigma^* \to \Sigma^*, \ \forall \omega, \ r(\omega) = t(\langle R \rangle, \omega)$

minimal description of TM MIN_{TM} fixed point $\exists F, \ f(\langle F \rangle) = F$

mathematical logic: model, formula ⊃ sentence ⊃ theory

$Th(\mathbb{N}, +)$ is deci. *proof: NDA recursion* $Th(\mathbb{N}, +, \times)$ is undeci.

oracle TM $T^{A_{TM}}$ is much stronger but there still be some lang. it can not deci.

A decidable relative to B Turing reducible $A \leq_T B$ *example:* $E_{TM} \leq_T A_{TM}$

5.5

minimal description of string $d(x)$, descriptive(Kolmogorov) complexity $K(x) = \min |\langle M, \omega \rangle|$ where x is on the tape when M halts on the input ω (or $K(x) = \min\limits_{p:\mathcal{U}(p)=x} l(p)$)

$K(x)$ is uncomputable. Godel incompleteness theorem, Berry paradox

$K(xy) \leq K(x) + K(y) + O(logK(x))$ but can not reach $K(x) + K(y) + O(1)$.

$\forall \text{ desc. lang. } A, \ \exists c_A, \ \forall x, \ K(x) \leq K_A(x) + c_A$

$|\{x \in \{0,1\}^* : K(x) < k\}| < 2^k$ for integer n, $K(n) \leq log^* n + c$

$\forall \mathcal{U}, \ \sum\limits_{p:\mathcal{U}(p)\text{halts}} 2^{-l(p)} \leq 1$ (by Kraft ineq.) $sto. \{X^n\} \overset{iid.}{\sim} f(x), \ \frac{1}{n} EK(X^n|n) \to H(X)$ (by source coding th.)

more: c-compressible There always exists incompressible string of any length.($\lim\limits_{n\to\infty} \frac{K(x^n|n)}{n} = 1$) There exists a constant b for $\forall x, d(x)$ is incompressible by b.

universal prob. $P_{\mathcal{U}}(x) = \sum\limits_{p:\mathcal{U}(p)=x} 2^{-l(p)} \ \forall \text{ computer } A, \ \exists c_A, \ \forall x, \ P_{\mathcal{U}}(x) \geq c_A P_A(x)$

equivalence: $\exists c, \ \forall x, \ |log \frac{1}{P_{\mathcal{U}}(x)} - K(x)| \leq c$

Chaitin $\Omega = \sum\limits_{p:\mathcal{U}(p)\text{halts}} 2^{-l(p)}$

more: Kolmogorov structure function, Kol. minimal sufficient statistics

6 Complexity

6.1

big O notation, small O notation

Unlike compuability, complexity depends on the computing model.

time complexity $TIME(f(n))$ *note:* $TIME(O(nlogn)) \subset REG$

$P = \bigcup\limits_k TIME(n^k)$ PATH, REL_PRIME, CFL

verifier, certificate $NP = \bigcup\limits_k NTIME(n^k) = P\_VERI \subset EXPTIME$ HAM_PATH, COMPOSITES, CLIQUE

$P \overset{?}{=} NP$ NP-complete SAT

polynomial time computable function, polynomial time reduction $A \leq_P B$

more: cnf formula, 3SAT(is NPc)

Cook-Levin Th: SAT is NPc. *proof: tableau, window* $\phi_{cell} \wedge \phi_{start} \wedge \phi_{move} \wedge \phi_{accept}$

6.2

space complexity $SPACE(f(n))$

$SPACE(f(n)) \subset TIME(2^{O(f(n))}) \subset SPACE(2^{O(f(n))})$

Savitch Th: $\forall f : \mathbb{N} \to \mathbb{R}^+, \text{ where } f(n) \geq n(\text{acc. } logn), \ NSPACE(f(n)) \subseteq SPACE(f^2(n))$

$PSPACE = NPSPACE \supseteq NP$

PSPACE-complete PSAPCE-hard TQBF, FORMULA_GAME

bitape TM $L = SPACE(logn) \stackrel{?}{=} NL$

log space transducer, log space reduction $A \leq_L B$

PATH is NLc, $NL = coNL \subseteq P$

6.3

space constructible($f(n) \geq O(logn)$), time constructible($f(n) \geq O(nlogn)$)

Hierarchy Th:

$$\forall \text{ constructible } f : \mathbb{N} \to \mathbb{N}, \ \exists \text{ lang. } A,$$
$$\text{decidable in space } O(f(n)) \text{ but not } o(f(n)); \text{ in time } O(f(n)) \text{ but not } o(f(n)/logn)$$

$NL \subsetneq PSPACE$, $PSAPCE \subsetneq EXPSPACE$, $P \subsetneq EXPTIME$

more: EXPSPACE-complete circuit complexity

advanced topics: approx. algorithm, probabilistic TM (BPP), prime, alternating TM, IP=PSPACE, parallel RAM (NC), cryptography(private-key cryptosystem, pulic-key cryptosystem RSA)

family picture: