0 集合、映射 群、群同态、有限生成的Abel群、自由群

1 点集拓扑

1.1 拓扑空间 开闭集/邻域/闭包 子空间拓扑 拓扑基

1.2 连续映射 同胚 积空间、商空间(粘合) 收敛性

1.3 可数性、分离性 Urysohn引理、Tietze扩张

1.4 紧性 紧化 连通性 道路连通、局部连通

1.5 度量化定理 仿紧、流形、函数空间

2 基本群

2.1 同伦 基本群及其计算 Van kampen定理

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3.6 局部同调群 棱道群 Jordan曲线、闭曲面分类

4 进一步论题

相对同调、上同调、奇异同调、同伦群、范畴

基础拓扑纲要
Basic Topology Outline

La.

$(f\cup g)^{-1}(C) = f^{-1}(C)\cup g^{-1}(C)$.

$j: \oplus X_\alpha \to \cup X_\alpha$. s.t. on X_α is including map.

$F: \cup X_\alpha \to Z$, $F(x) = f_\alpha(x)$ if $x\in X_\alpha$.

$f_\alpha : X_\alpha \to Z$ s.t. $f_\alpha | X_\alpha\cap X_\beta = f_\beta | X_\alpha \cap X_\beta$.

(连续、满) (cont. surj.) onto. 若 X (X_α) 与 Y 皆为 X∪Y ($\cup X_\alpha$) 的开/闭集，则 j 为商射.

若 j 为商射，则 f_α 连续 => F 连续. (since $F_j : \oplus X_\alpha \to Z$ cont. <=> f_α cont.)

△ 粘射不一定是开射. (not all $V\in\tau$ can be written as $f^{-1}(U)$)

($\cup X_\alpha$ 的商拓扑: $A\in\tau \iff A\cap X\in\tau_X,\ A\cap Y\in\tau_Y$.)

△ 若 α 无限需注意 总集给以 $\cup X_\alpha$ 的拓扑可能与商/粘合给以的不同.

ex. 

$m: G\times G\to G$, $i: G\to G$ cont.

ex. $GL(n)$. (给GL(n)以 $M\subset E^{n^2}$ 诱导的子拓)

$O(n)$、$SO(n)$ comp.

$O(n-1)\cong O(n)$ 的一个子群.

$SO(2)\cong S^1$, $SO(3)\cong P^3$. (S^3 双倍覆盖)

ex. $\frac{\mathbb{R}}{\mathbb{Z}}\cong S^1$. 同胚&群同构.

$f:\mathbb{R}\to S^1$ induces quo. map.

齐性拓扑空间 (局部拓扑结构相似) $\forall x,y\in G$, $\exists$ mor. f $f(x)=y$ (i.e. $L_{yx^{-1}}$)

K 为 G (含e) 的连通分支 => 闭正规子群. $KK^{-1}=K$, $gKg^{-1}\subset K$

$G=O(n)$, $K=SO(n)$.

△ $\det: M\to\mathbb{R}$

thus $GL(n)$ not comp. ($\in\tau_M$) 

not connec. ($\det>0$, $\det<0$) only for $GL(n,\mathbb{R})$.

极大紧子群 $O(n)$.

Pf: if $O(n)\subsetneq K$, let $A\in K\setminus O(n)$,

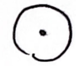
$\exists x\in\mathbb{R}^n$, $\|x\|=1$, s.t. $\|Ax\|\neq 1$, $\|Ax\|\neq 0$,

$\exists Q\in O(n)$, s.t. $Qx=\frac{Ax}{\|Ax\|}$.

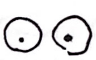
$\|Ax\|$ ($\neq 1$) is a eigenval of $Q'A$. (e!, eigenval must be 1.)

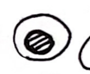
thus $Q'A\notin K$, ✗.

△ 分离公理 单点集(<=>有限集)闭.

T_0:  Kolmogorov.

T_1:  Frechet.

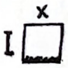
T_2:  Hausdorff → 序列收敛值唯一.

T_3:  reg. Haus.

T_4:  normal Haus. → Urysohn, Tietze.

1,2,3 ✓ 遗传、可乘 4 ✗.

△度量化定理 T_1 T_4 C_2.

ex. I  X $CX\cong$ 几何 cone. (two way: $\frac{X}{\sim}$ or $\frac{X}{A}$)

$\frac{D^n}{S^{n-1}}\cong S^n$. ($\cong\partial D^n$)

△ 四元数 $\mathbb{H}$, 欧拉角.

$\frac{X}{G}$ (可迁作用) ($x\sim g(x)$)

$\frac{S^{n-1}}{O(n)}\cong\{0\}$.

$\frac{O(n)}{O(n-1)}\cong S^{n-1}$.

induces a mor. of $A\mapsto A\begin{pmatrix}1&0\\0&B\end{pmatrix}$.

(or quo. map $f: O(n)\to S^{n-1}$ induced partition. $f(A)=Ae_1$).

$\pi: X \to \frac{X}{G}$ is open.

G. $\frac{X}{G}$ connect. $\Rightarrow$ X connect.

ex. dense true subset orbit: (环面上的无理流)

$\mathbb{E}^2 \xrightarrow{\pi} \frac{\mathbb{E}^2}{\mathbb{Z}\times\mathbb{Z}} \cong S^1 \times S^1$ orbit space.

$(x,y) \to (e^{2\pi i x}, e^{2\pi i y})$

△ Plane Symmetry groups.
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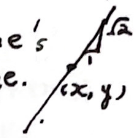
$\mathbb{R}$ induces a mor. of $r: (e^{2\pi i x}, e^{2\pi i y}) \to (e^{2\pi i (x+r)}, e^{2\pi i (y+\sqrt{2} r)})$.

$O(x,y)$ is the line's image.



$\sqrt{2}$, (x,y)

0, 1, 1

$f \underset{F}{\simeq} g$ rel. A $\{0,1\}$

$f, g: I \to X$

str.-line mot. $f, g: X \to C$,

$F: X \times I \to C$ convex set.

$F(x,t) = t g(x) + (1-t) f(x)$

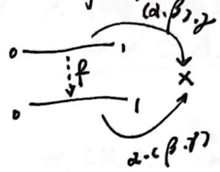
△ ant. $f: X \to S^n$ not surj. $\Rightarrow$ f null-mot.

$\Leftrightarrow$ f can be expanded onto $CX \to Y$. $\left(\frac{X \times I}{X \times \{1\}}\right)$

$(\alpha\cdot\beta)\cdot\gamma \simeq \alpha\cdot(\beta\cdot\gamma)$

thus $\langle\alpha\rangle\cdot\langle\beta\rangle = \langle\alpha\cdot\beta\rangle$ is a group.

(mot. class of α: $\langle\alpha\rangle$)



$(\alpha\cdot\beta)\cdot\gamma$, f, x, $\alpha\cdot(\beta\cdot\gamma)$

$(\alpha\cdot(\beta\cdot\gamma))$ of (compared)

$\pi_1(X,p) \xrightarrow{\gamma^*} \pi_1(X,q)$ i.e. $\cong$. so $\pi_1(X)$.

$p \xrightarrow{\gamma} q$ $\quad \langle\alpha\rangle \longmapsto \langle\gamma^{-1}\cdot\alpha\cdot\gamma\rangle$ (bij.)

$\mathbb{R}^n$ / Convex $C\mathbb{E}^n$ / 单连通 $\to$ trivial.

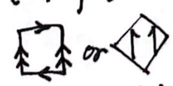
Mobius / $I \times S^1$ / S^1 $\to \mathbb{Z}$
(cb)

$(n \geq 2)\ S^n \to$ tri.

$S^1 \times S^1 \to \mathbb{Z}\times\mathbb{Z}$

$(n \geq 2)\ P^n \to \mathbb{Z}_2$

$L(p,q) \to \mathbb{Z}_p$.



or Klein($\mathcal{K}$) $\to \{a, b \mid a^2 = b^2\}$.
(or $tut = u$, $\{t, u\}$)

cont. $f: X \to Y$,

$f\circ(\alpha\cdot\beta) = (f\circ\alpha)\cdot(f\circ\beta)$

$(f\circ g)_* = f_* \circ g_*$

mor. f_* induced by f $\quad f_*(\langle\alpha\rangle) = \langle f\circ\alpha\rangle$,

$f_*: \pi_1(X) \to \pi_1(Y)$.
$\quad p \quad\quad q \quad (q = f(p))$

同胚的(道连)空间具有同构的基本群.

$X \underset{g}{\overset{f}{\rightleftarrows}} Y$ $\quad g f \simeq 1_X$, $f g \simeq 1_Y$.

mot. eq. $X \simeq Y$
(form)

△ 同构三定理:

1. $\sigma: G \to G'$ is a morphism.
$N = \ker(\sigma)$ is normal subgroup,
then $\frac{G}{N} \cong \sigma(G)$.

2. $H \supset N$ normal subgroup,
then $\frac{G}{H} \cong \frac{\sigma(G)}{\sigma(H)}$.

3. $H \subset G$ is subgroup, then
$H \cap N$ is H's normal subgroup,
$\frac{H}{H\cap N} \cong \frac{HN}{N}$.

ex. S^1's basic group $\mathbb{Z}$:

'descend' $\pi: \mathbb{R} \to S^1$, $x \mapsto e^{2\pi i x}$.

lifted roads $\gamma_n(s) = ns$, $s \in [0,1]$. (Lifting Lemma)

$\pi\circ\gamma_n$ is the ring roads.

$\phi: \mathbb{Z} \to \pi_1(S^1, 1)$ is mor.
$n \mapsto \langle\pi\circ\gamma_n\rangle$.

- morph. $\phi(m+n) = \phi(m)\cdot\phi(n)$.
- surj. lift the road: $\pi\circ\tilde{\sigma} = \sigma$.
- bij. $\Leftarrow \ker(\phi) = 0$ lift the mot.: $\pi\circ\tilde{F} = F$

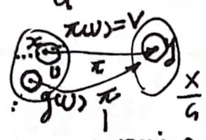
△ $X = U \cup V$, U、V 单连通且 $U \cap V$ 道路连通,
则 X 单连通.

covering space $\pi: X \to Y$ $\forall y \in Y, \exists$ nei. V, a partition of $\pi^{-1}(V)$ i.e. $\{U_\alpha\}$.
(say, maybe smaller fun. grp.) s.t. $U_\alpha \underset{\pi}{\cong} V.\ (\forall \alpha)$. (multiplicity. e.g. roots)

mor-grp. G on simply-connect X, $\forall x \in X\ \exists$ nei. U s.t. $U \cap g(U) = \phi$ for $\forall g \in G - \{e\}$
then $\pi_1(\frac{X}{G}) \cong G$. (=> G discrete topo)

($\pi: X \to \frac{X}{G}$ is covering map. Def: $\phi: G \to \pi_1(\frac{X}{G}, \pi(x_0))$
$g \mapsto \langle \pi \circ \gamma \rangle$
road $\gamma: x_0 \rightsquigarrow g(x_0)$.)

X, Y pathconn. sp.,
$\pi_1(X \times Y) \cong \pi_1(X) \times \pi_1(Y)$.



covering.

mot. form: $\mathbb{E}^n - \{0\} \simeq S^{n-1}$

Convex $\simeq \{0\}$
$C/\mathbb{E}^n$

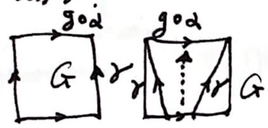
retraction
X 形变收缩到A
$\Rightarrow X \simeq A$.

$\simeq 8$ or 8 or θ.
fungrp: $\mathbb{Z} * \mathbb{Z}$.

$f \underset{F}{\simeq} g: X \to Y$, f_*与g_*只差一个同构.

$\pi_1(X, p) \xrightarrow{g_*} \pi_1(Y, g(p))$.
i.e. $\pi_1(X, p) \xrightarrow{f_*} \pi_1(Y, f(p)) \xrightarrow{\gamma_*} \pi_1(Y, g(p))$.
where $\gamma_*: \langle \alpha \rangle \mapsto \langle \gamma^{-1} \cdot \alpha \cdot \gamma \rangle$. γ is the path from $f(p)$ to $g(p)$.
$\gamma: f(p) \rightsquigarrow g(p)$

Pf: 证 $g \circ \alpha$ mot. $(\gamma^{-1} \cdot (f \circ \alpha)) \cdot \gamma$
Let $G(s,t) = F(\alpha(s), t)$.
$G: I \times I \to Y$



contractible: $1_X \simeq C_p$ (null-mot.)
$\gamma: p \rightsquigarrow g \circ f(p)$

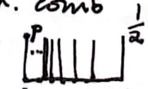
path. conneet.
+
mot. form. => mor. fungrp. Pf:

$X \underset{g}{\overset{f}{\rightleftarrows}} Y$, p, $f(X)$, $g(Y)$

证 $f_*: \pi_1(X, p) \to \pi_1(Y, f(p))$ 同构.
bij. $(g \circ f)_* = \gamma_* \circ 1_{X*} = \gamma_*$ mor. ✓
surj. $(f \circ g)_*$ similarly ✓

ex.
$\mathbb{E}^n - \{0\} \simeq S^{n-1}$ Mo. $I \times S^1$.
$n \geq 3$ 单连通. $\mathbb{E}^2 - \{0\} \simeq S^1$
so $\mathbb{Z}$.

ex. 'comb'



$1_X \simeq C_p$ rel. $\{p\}$

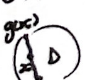
$\triangle$ $X_i \simeq Y_i$, $X_1 \times X_2 \simeq Y_1 \times Y_2$.
$i = 1, 2$.

$\triangle$ $X \simeq Y$, X path conn <=> Y pathconn.

$\forall X$, CX contractible.

Brower fixedpoint. $n=1$: $I = \{x \in I \mid f(x) < x\}$ is not exist. (强形变收缩核)
$\cup \{x \in I \mid f(x) > x\}$
or $g: I \to \{0,1\}$ cont.
but I connected, $\{0,1\}$ not.

$n=2$:

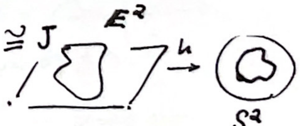


morph. g.
cont. $f(x)$
$g_*: \pi_1(D) \to \pi_1(C)$
is surj. but $\pi_1(D) \cong \{0\}$,
$\pi_1(C) \cong \mathbb{Z}$.
i.e.
(收缩映射诱导了基本群的满同态)

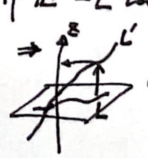
$\forall n$: g cont. & surj.
$\Rightarrow g_*$ surj. morph.
$B^n \to S^{n-1}$

↓
同调群 $H_{n-1}()$
$0 \neq \mathbb{Z}$

$S^1 \cong J$
Joran curve.



Pf: if $\mathbb{E}^2 - L$ path connect $\Rightarrow \mathbb{E}^3 - L$ simply connect



$\mathbb{E}^3 - (\text{axis} B)$ simply connect.
$\simeq \mathbb{E}^2 - \{0\}$

thus $\pi_1(\mathbb{E}^3 - L) \cong \pi_1(\mathbb{E}^2 - \{0\}) \cong \mathbb{Z}$
X.

Surface
$\mathring{S}$: ∂S
$\mathbb{E}^2 \cong V$: $\mathbb{E}^2_+ \cong U$

mor.
$h: S_1 \to S_2$,
then $h(\partial S_1) = \partial S_2$,
$h(\mathring{S}_1) = \mathring{S}_2$.
ex.
Mo $\ncong$ $I \times S^1$ (柱面)

Simplical Complex K $\quad (|K| \cong X)$ $\quad$ cones: $|CK| \cong C|K|$

Subdivision $\overset{\text{mor.}}{h} : |K| \to X$. 复形同构给出多面体同胚.

$|K|$ compact. connect $\overset{}{\Longleftrightarrow}$ path-connect. $\quad$ barycentric subdivision K^m.

$\subset E^n$ (closed + bounded)

(由各单纯形 $A \in K$ 取并、粘合得到$|K|$.) 闭 ⇒ 闭.

simplical approximation Th. $\exists m$, (dim $s(A) \leq$ dim A) $s \simeq f$. simp. $s: |K^m| \to |L|$ approx. $f: |K^m| \to |L|$. cont.

(f^{-1} 分别应)

edge group $E(K, v) \cong \pi_1(|K|, v)$.

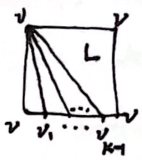
Pf: morph. + surj. + bij.

simp. map. 保持棱道间的等价.

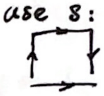
证 edge $v \ldots v$ 映入$|K|$内时零伦 ⇒ $v \ldots v$ 等价于v.

group presentation $G = \{S|R\}$, $G \cong \frac{F(S)}{N}$. $\quad$ R gens.

$F: \underbrace{I \times I}_{|L|} \to |K|$ $\quad$ simply-conn.



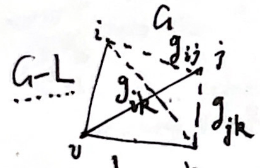
use $s: |L^m| \to |K|$ approx. F,



equivalance.

$G(K, L) \cong E(K, v)$

(v in max tree simp. subcomplex L)



ex. circle bundle

$\pi_1(X) = \underbrace{\mathbb{Z} * \mathbb{Z} \cdots}_{\text{free grp of } n\text{'s generator}}$

thus $\forall$ path-connected subdivisionable sp. has finite presentation basic grp.

△ Cayley graph:

$\pi_1(|K|) \cong \pi_1(|K(2)|)$. '2 bonds'.

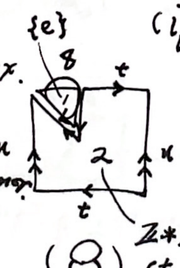
van Kampen Th: $\pi_1(X_1 \cup X_2, v) \cong \pi_1(X_1, v) * \pi_1(X_2, v) / \{i_{1\pi}(\alpha) i_{2\pi}(\alpha^{-1}) \mid \alpha \in \pi_1(X_1 \cap X_2, v)\}$.

(X_1, X_2, $X_1 \cap X_2$ pathconn.)

(here can use $|K_1|$, $|K_2|$ for complex.)

($i_{l\pi}$ is the induced mor. of including map $i_l : X_l \to X_1 \cup X_2$, $l = 1, 2$)

ex.



$i_{1\pi}(8) = i_{2\pi}(8)$ means $e = tu^{-1}tu$

thus $\mathcal{K}$'s $\{t, u \mid tut = u\}$.

vertex form $K = \{V, S\}$

(同胚)群的单纯作用 $\pi^{-1}\phi\pi$ is simp. (subdivision mor. $\pi: |K| \to X$)

⇒ $g \in G$ induces $\pi^{-1}g\pi : |K| \to |K|$ a self-mor.

inf. complex.

(不一定紧但局部紧)

$$\begin{tikzcd} {|K^2|} \arrow[r, "\pi"] \arrow[d, "s"] & X \arrow[d, "i"] \\ {\left|\frac{K^2}{G}\right|} \arrow[r, "h\psi^{-1}"] & \frac{X}{G} \end{tikzcd}$$

(保持单纯形 $p(v_i) = u_i$)

$h: \frac{|K|}{G} \to \frac{X}{G}$ is the induced mor. of orbit sp. of π.

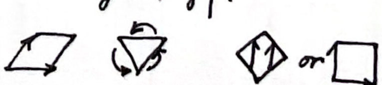
$$\begin{tikzcd} {|K|} \arrow[d, "p"] \arrow[dr, "s"] & \\ \frac{|K|}{G} \arrow[r, "\psi"] & {\left|\frac{K}{G}\right|} \end{tikzcd}$$ proj.

since G simp. on X, simp. map s has $sg(x) = s(x)$, thus induces $\psi: \frac{|K|}{G} \to \left|\frac{K}{G}\right|$. when K^2, ψ bij. mor.

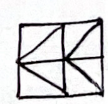
$h\psi^{-1}$ is a subdivision of orbit sp. $\frac{X}{G}$.

Edge grp still $\cong$ basic grp, but maybe not finite-presentation.

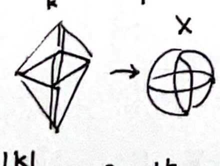
ex. crystal grp.



tores T^2 $\quad$ sphere S^2 $\quad$ Klein $\mathcal{K}$.



ex.



$G = \mathbb{Z}_2$.

$\frac{|K|}{G} \cong P^2$, $\left|\frac{K}{G}\right| \cong D^2$.

△ N normsub. of G.

$$\frac{X}{G} \cong \frac{\frac{X}{N}}{\frac{G}{N}}.$$

△ X simplyconn.

G simp. on X, $\frac{X}{G} \cong \frac{G}{F}$. where F is normsub. that has at least one fixedpoint in X.

(Simpilical) Triangulation.

▵ closed surface: compact. connected. no boundary. nei. $\cong \mathbb{E}^2$. $\Rightarrow$ closed manifold. $L(p,q)$ 闭3流形

(Rado.) every comp. surface can be triangulated. combinatorial surface K.

Orientation. incrassate L: Cylinder or Moebius. $N(L)$ $(\bigcup_{star}(v, K^2))$ $\chi(K) \leq 2$. $(=\chi(T)+\chi(\Gamma))$

Euler characteristics $\chi(L) = \sum_{i=0}^{n} (-1)^i \alpha_i$. L is dim n finite simp. complex, α_i is the number of i-simplex in L.

graph Γ, $\chi(\Gamma) \leq 1$. (=1 when Tree.)

dual graph $T \sim \Gamma$. Surgery (割补运算) (or say Γ' is all simplex disjoint with T in K'.) $|K|$等 $(N(T), N(\Gamma) \cong D^2$, thus S^2)

等价: $|K| \cong S^2$、$\chi(K)=2$、K' edge forming closed poly curve seperate $|K|$. Γ is tree

$\chi(L)$ is sp. $|L|$'s topological Const. Barycentric. keeps $\chi(K)$.

▵ G simp. on $|K|$, free. $\chi(K) = |G| \cdot \chi(\frac{K^2}{G})$. $\chi(K \cup L) = \chi(K) + \chi(L) - \chi(K \cap L)$.

(Pf: $=\chi(K^2)$, and $\forall$ simplex σ_i in $\frac{K^2}{G}$, corresponds $|G|\sigma_i$ in K^2.)

K surgery on $L \to K^*$ (L closed but not sep.) $(M = K - N + L_i)$
$= M \cup CL_1 \cup CL_2$ (Cylin)
(L thicken to N) $/= M \cup CL_1$ (Mo)
(L_1. L_2 is the boundary circle)

revert surgeries.

every closed surface $\cong$ one of standard closed surface. $\chi(N)=0$

(orientable $\times$ Mo) Cylin. $(\chi(L)=0, \chi(\partial L)=1) \Rightarrow \chi(K^*) > \chi(K)$.

Mo. (when have Mo or unorientable, Cylin(+2) equals 2∗Mo (+1).) $(mP^2 \# nT^2 \cong (m+2n)P^2)$ or

S^2
$H(p)$ genus $p,q \in \mathbb{N}^+$. $\{e\}$ $\prod_{i=1}^{p} a_i b_i a_i^{-1} b_i^{-1} = e$ (+2/+1) $\xrightarrow{\text{Abelize}} \mathbb{Z}\times\cdots\times\mathbb{Z}$ ($2p$)
$M(q)$ $\prod_{i=1}^{q} a_i^2 = e \xrightarrow{\text{Alize}} \mathbb{Z}_2 \times \mathbb{Z}\times\cdots\times\mathbb{Z}$ ($q-1$)

surface symbols: ($\mathcal{K}$)

▵ 一个紧连曲面完全取决于其定向性、亏格、边界圆周数.

ori. $a_1 b_1 a_1^{-1} b_1^{-1} \cdots a_p b_p a_p^{-1} b_p^{-1} = e$ ($4p$ polygon)

unori. $a_1 a_1 a_2 a_2 \cdots a_q a_q = e$ ($2q$ polygon)

note: 'ring handle' $\Rightarrow$ $\Rightarrow \mathbb{Z} * \mathbb{Z}$ $(aba^{-1}b^{-1}=e)$ (wedge sum) sp. ($\vee$ bouquet) # connected sum ex. $T^2 \# T^2$.

$(T^2 - p \simeq S^1 \vee S^1)$ (frame) $T^2 \Rightarrow \mathbb{Z}\times\mathbb{Z}$ (commutable)

Curves at surface piece's boundary maybe not null-mot.

▵ 闭曲面挖洞后可收缩为圆束, $k = 2p$ or q. (S^2, pT^2, qP^2) (k个S^1一点并)

Cycles & Boundaries

oriented simplex: $\sigma + \tau = 0$ $C_q(K)$ free Abelian grp. (of q-dim chains)

$(\partial_q : C_q \to C_{q-1})$
$\partial^2 \equiv 0$. i.e. $\mathrm{Im}(\partial_q) = B_{q-1}(K) \subset Z_{q-1}(K)$ bound. grp.

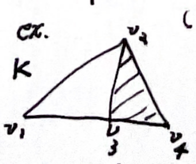
$\partial(v_0, \cdots, v_q) = \sum_{i=0}^{q} (-1)^i (v_0, \cdots \hat{v_i}, \cdots v_q)$ bound. morph. $\partial: C_q(K) \to C_{q-1}(K)$ its kernel $Z_q(K)$ (of closed q-dim chains) homo. grp. $H_q(K) = \frac{Z_q(K)}{B_q(K)}$ [8] homo. class.

$H_0(K)$ is free., rank = $|K|$'s conn. components num. (bound. of $C(K)$)

(if conn. K, $H_0(K) \cong \mathbb{Z}$) finite Abel grp.

ex.
Torus $H_2(T) \cong \mathbb{Z}$
Klein $H_2(\mathcal{K}) \cong 0$ (unori.)

ex. K



$H_q(K) = F \oplus T$ torsion ele.

if K is cone $(K \cong CL)$, simplex σ has $\sigma = \partial d(\sigma) + d\partial(\sigma)$ thus σ closed chain, $H_q(K) = 0$ for cones $(q>0)$. β_q Betti number. (dim q)

$(d: C_q(K) \to C_{q+1}(K)$, σ in L then adds vertex v, otherwise $d(\sigma)=0$.)

(in fina., $\oplus$ (inner) direct sum $\cong$ $\times$ (outer) Dic. prod.) prod.

$$H_q(S^n)=\begin{cases}\mathbb{Z} & q=0\\ \mathbb{Z} & q=n\\ 0 & \text{other } (q\le n-1,\ H_q(S^n)\cong H_q(\Delta^{n+1})=0 \text{ because of cone})\end{cases}\qquad H_q(D^n)=\begin{cases}\mathbb{Z} & q=0\\ 0 & \text{other.}\end{cases}$$

comb. surface K, $S^2\cong|K|$

$H_0(K)=\mathbb{Z}$, $H_1(K)=\begin{cases}0 & \\ 2g\mathbb{Z} & g\text{'s ori.}\\ (g-1)\mathbb{Z}\oplus\mathbb{Z}_2 & g\text{'s unori.}\end{cases}$

$H_2(K)=\begin{cases}\mathbb{Z} & \text{ori.}\\ 0 & \text{unori.}\end{cases}$ Δ

if $|K|$ conn., (basic grp) Abelize to $H_1(K)$.

$\pi_1(K)$ $\dfrac{\pi_1}{[\pi_1,\pi_1]}\cong H_1(K)$ (first-order homo. grp)

Pf: ($Z_1(K)$可由初等1闭链即(定向)简单棱环道生成)

morph. $\phi: E(K,v)\longrightarrow H_1(K)$

ϕ surj., $\ker(\phi)$ is $E(K,v)$'s commutator subgrp.

$S_q(-\sigma)=-S_q(\sigma)$

induce: ($S_q(\sigma)=0$ iff $s(v_i)$有相同点)

$s\to S_q\longrightarrow S_q^*$

($s:|K|\to|L|$ simp.) since

$$\begin{array}{ccc} \vdots & & \vdots\\ \downarrow\partial & & \downarrow\partial\\ C_q(K) & \xrightarrow{S_q} & C_q(L)\\ \downarrow\partial & & \downarrow\partial\\ C_{q-1}(K) & \xrightarrow{S_{q-1}} & C_{q-1}(L)\\ \vdots & & \vdots\end{array}$$

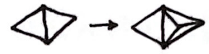
$(\psi\circ\phi)_*=\psi_*\circ\phi_*$.

(chain map $\psi.\phi$, induce morph. $\psi.\phi: C(K)\to C(L)$ in homo. grp $\psi_*.\phi_*$.)

$\partial S_q=S_{q-1}\partial: C_q(K)\to C_{q-1}(L)$

$S_q(Z_q(K))\subset Z_q(L)$, $S_q(B_q(K))\subset B_q(L)$

Stellar subdivision $\Rightarrow$ Barycentric subdivision



keeps K's homo. gps.

($\chi: C(K)\to C(K^{m})$ induces mor.)

subd. chain map. $\chi_*: H_q(K)\to H_q(K^{m})$

($\theta: |K^{m}|\to|K|$ std. simp. map)

$\forall$ cont. $f:|K|\to|L|$ induces morph. $f_*: H_q(K)\to H_q(L)$. (Pf: simp. approx. $f_*=S_*\circ\chi_*$)

$|K|\xrightarrow{f}|L|\xrightarrow{g}|M|$, then $(g\circ f)_*=g_*\circ f_*: H_q(K)\to H_q(M)$; iden. $1:|K|\to|K|$ then 1_* iden.

$f\simeq g:|K|\to|L|$, then $f_*=g_*: H_q(K)\to H_q(L)$.

$$\left(\text{Pf: }\begin{array}{ccccc} H_q(K^m) & \xrightarrow{S_*} & H_q(L^n) & & \\ \chi_{1*}\uparrow\downarrow\theta_* & & \chi_{2*}\uparrow\downarrow\theta_{2*} & \searrow t_* & \\ H_q(K) & \xrightarrow[f_*]{} & H_q(L) & \xrightarrow[g_*]{} & H_q(M)\end{array}\right)$$

$\Longrightarrow$ $|K|\simeq|L|$, then $H_q(K)\cong H_q(L)$. (so $\forall t:|K|\to X$ triangulates X has the same homo. grp)

(proof of $\mathbb{E}^m\ncong\mathbb{E}^n$: if $\cong$, $S^{m-1}\simeq\mathbb{E}^m-\{o\}\cong\mathbb{E}^n-\{o\}\simeq S^{n-1}$,

$H_{m-1}(S^{m-1})\cong\mathbb{Z}\cong H_{m-1}(S^{n-1})$ iff. $m=n$.)

cont. $f:S^n\to S^n$, $h:|K|\to S^n$. $f^h=h^{-1}fh:|K|\to|K|$ induces $f_*^k: H_n(K)\longrightarrow H_n(K)$, $f_*^k(z)=\deg f\cdot z$.

(inf. cyclic grp. $(\cong\mathbb{Z})$ generator $[z]$) $\to\lambda[z]$.

$\deg f=\deg g\Leftrightarrow f\simeq g$. ex. deg: mor. ±1; 1_X 1; C_p 0

$\deg f\circ g=\deg f\cdot\deg g$.

ex. $v_1.v_{-1}\dots v_{n+1}.v_{-(n+1)}\to\Sigma$ (面$(v_{i_1}\dots v_{i_s})$ 且 $|i_1|<\dots<|i_s|$) antipodal. $(-1)^{n+1}$

subd. $\pi:|\Sigma|\to S^n$

($|\Sigma|$ 即 $\sum_{i=1}^{n+1}|x_i|=1$ 区域)

simp. $s:|\Sigma^m|\to|\Sigma|$ approx. f^π.

$f_*^\pi: H_n(\Sigma)\to H_n(\Sigma)$

$\sigma=(v_1,\dots,v_{n+1})$.

$\alpha=$ number of n-simplex τ s.t. $s(\tau)=\sigma$ in Σ^m

$\beta=\dots$ s.t. $s(\tau)=-\sigma$. $\Rightarrow\deg f=\alpha-\beta$.

(cont. $f:S^n\to S^n$ without fixpoint must have deg $(-1)^{n+1}$. Pf: mot. $F=\dfrac{(1-t)f(x)-tx}{\|(1-t)f(x)-tx\|}$

(thus if $f\simeq 1_{S^n}$, $F:S^n\times I\to S^n$.)

(otherwise has antipodalpoint) Δ n even, then f has fixpoint.)

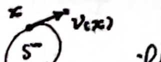
Δ n even, only $\mathbb{Z}_2$ and $\{e\}$ can act freely on S^n.

($\forall\mathbb{Z}_p$ can free on S^3)

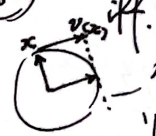
(if cont. f $\deg\neq1$, then f has antipodalpoint.)

Pf: $g\circ f$, g is anti. map.

Δ n even: no cont. vec. field exists (without $v(x_0)=x_0$). cont. $f=\dfrac{v(x)}{\|v(x)\|}\simeq1$, thus fixpoint.



iff. n odd there can be a cont. vec. field on S^n without zero.



$x=(x_1,\dots,x_{2m})\longrightarrow v(x)=(x_1-x_{m+1},\dots,x_m-x_{2m},x_{m+1}+x_1,\dots,x_{2m}+x_m)$

'hairball'

Euler-Poincaré $\chi(K)=\sum_{q=0}^{n}(-1)^q\beta_q$.   ($n=\dim K$)   (vec. sp. on field $\mathbb{Q}$)

(Pf: $\alpha_q=\dim C_q(K,\mathbb{Q})$. $\beta_n=\dim H_q(K,\mathbb{Q})$ since $[w]$ has $mw=$ bound. $w=\frac{1}{m}$ bound, $\{w\}=0$)

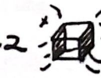
$$\underbrace{\underbrace{\underbrace{\partial c_1^{n+1},\dots,\partial c_{\gamma_{n+1}}^{n+1}}_{B_n\text{ basis}},\ z_1^n,\dots,z_{\beta_n}^n}_{Z_n\text{ basis}},\ c_1^n,\dots,c_{\gamma_n}^n}_{C_n\text{ basis}}$$

thus $\chi=\sum_{q=0}^{n}(-1)^q(\gamma_{q+1}+\beta_q+\gamma_q)$.

($\gamma_{n+1}=\gamma_0=0$)

ex. surface $\begin{cases}\text{ori. } \chi=2-2g\\ \text{unori. } \chi=2-g\end{cases}$.

△ $\chi(X\#Y)=\chi(X)+\chi(Y)-2$



△ $\chi(|K|\times|L|)=\chi(|K|)\cdot\chi(|L|)$

(Künneth Formula: $H_n(X\times Y)=\bigoplus_{p+q=n}H_p(X)\otimes H_q(Y)$)

△ $\chi(T^n)=0$.

△ 'mod 2' chain. for closed surface Σ, $H_n(\Sigma,\mathbb{Z}_2)\cong\mathbb{Z}_2$. ($\chi(K)=\sum_{q=0}^{n}(-1)^q\beta_q^{\mathbb{Z}_2}$ ✓)

(K 的以(交换群) G 为系数的同调群)

cont. $f:S^n\to S^n$ keeps antip., deg f odd.
($\forall x\in S^n$, $f(-x)=-f(x)$)

$\Rightarrow$ $f:S^m\to S^n$ keeps antip., $m\leqslant n$.

$\Rightarrow$ (Borsuk-Ulam) $f:S^n\to\mathbb{E}^n$ must map an antips to one point.

(Pf: if $f(x)\neq f(-x)$, $\forall x$, $g(x)=\dfrac{f(x)-f(-x)}{\|f(x)-f(-x)\|}:S^n\to S^{n-1}$ keeps antip)

(thus S^n cannot nested in $\mathbb{E}^n$)

(Lusternik-Schnirelmann)

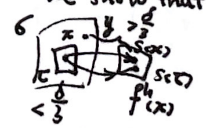
$S^n=\bigcup_{i=1}^{n+1}A_i$, A_i closed, $\exists A_{i_0}$ contains a pair of antips. $\Leftarrow$ (Pf: $f(x)=(d(x,A_1),\dots,d(x,A_n))$ thus $\exists y\in S^n$, $d(y,A_i)=d(-y,A_i)$ $1\leqslant i\leqslant n$. $=0$ then $y\in A_i$; >0 then $y\in A_{n+1}$)

Lefschetz number $\Lambda_f=\sum_{q=0}^{n}(-1)^q\operatorname{tr}f_{q*}$

(Λ_f 与 $h:|K|\to X$ 选取无关, ($H(K)\cong H(\mathbb{Q})$)
同态 $f_{q*}^h:H_q(K,\mathbb{Q})\to H_q(K,\mathbb{Q})$ 为 linear map)

$f\simeq g$, then $\Lambda_f=\Lambda_g$.   If $\Lambda_f\neq0$, f has fixed point. (Pf: simp. $s:|K^m|\to|K|$ approx. $f^h:|K^m|\to|K|$.

since Hopf, 证 $s_q\chi_q:C_q(K,\mathbb{Q})\to C_q(K,\mathbb{Q})$ trace $=0$.
if no fixpoint, $\delta=\operatorname{Inf}d(x,f(x))>0$.
we show that τ in $\chi_q(\sigma)\to s(\tau)\neq\sigma$



$s(x)$, y not in same.

thus $\operatorname{tr}(s_q\chi_q)=0$.)

Hopf trace Th. $f_q:C_q(K,\mathbb{Q})\to C_q(K,\mathbb{Q})$ (esp. simp if no fixpoint, then complex $A\neq f(A)$, thus trace $=0$)

$$\sum_{q=0}^{n}(-1)^q\operatorname{tr}f_q=\sum_{q=0}^{n}(-1)^q\operatorname{tr}f_{q*}.$$

$\Lambda_f=1+(-1)^n\deg f$, for cont. $f:S^n\to S^n$.   △ $\chi(S^n)=1+(-1)^n$.

$\chi(X)=\Lambda_{1_X}$. thus if $1_X\simeq$ no fixp. map, $\chi(X)=0$.
(in surface, only Torus and Klein)

compact. tria. sp has fixp property if has same H with a point (every self cont. f has fixp)
i.e. $H_0(K,\mathbb{Q})\cong\mathbb{Q}$; $H_q(K,\mathbb{Q})=0$, $q>0$.

↓

(Brawer)

$\forall$ comp. tria. contractible sp. has fixp property.
(not only B^n, but also e.g. P^n)

△ $f:X\to X$ null-mot., then it has fixed point.

covering dimension: (open cover $\mathfrak{F}$)

dim Compact. sp. $X=\sup_{\mathfrak{F}}D(\mathfrak{F})$,

$D(\mathfrak{F})=\inf_{\mathfrak{F}_*}\dim\mathfrak{F}_*$, $\mathfrak{F}_*$ is the refinement of $\mathfrak{F}$.

$\dim\mathfrak{F}_*$ is the dim of its complex (defined by $\{\mathfrak{F},S\}$) 'nerve' of $\mathfrak{F}$.

$\dim K=m\Rightarrow\dim|K|=m$.

$X\cong Y\Rightarrow\dim X=\dim Y$.

$Y\subset X\Rightarrow\dim Y\leqslant\dim X$

summary of S^n map: ① antipodal $\deg=(-1)^{n+1}$.
② $\Lambda=1+(-1)^n\deg$; $\Lambda\neq0\Rightarrow$ has fixed point.

$\Rightarrow$ n even,
$\deg=-1$ no fixed point;
at least fixed point or antipodal point;
(非0 vec. field) hair can't comb smoothly.

isotopy $H: X \times I \to Y$ for $\forall t \in [0,1]$, $H(t)$ is a mor. $X \underset{H(t)}{\cong} Y$.

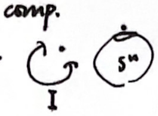
△ G is comp. conn. tria. topo. group, $\chi(G) = 0$

Torus is the only closed surface that is a topo. group.
(of S^n, only S^1 and S^3 can be)

△ X Hausdorff, locally comp., (X, τ)

$\tilde{X} = X \cup \{\infty\}$, $U \in \tilde{\tau}$ iff. $\begin{cases} \infty \notin U: U \in \tau \\ \infty \in U: \tilde{X} - U \text{ comp.} \end{cases}$

$(\tilde{X}, \tilde{\tau})$ is the one-point compactification.

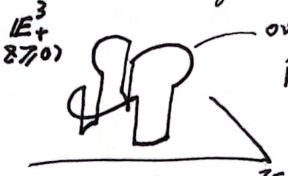


polygonal knot (tame knot)
nice proj. — countable inf.

ex. $h_t: \mathbb{E}^3 \to \mathbb{E}^3$ induces $\hat{h}_t: S^3 \to S^3$. keep ori. $\simeq 1$.

if $h_0 = 1$, $h_1 = h$, $h(k_1) = k_2$, k_1 equals k_2.

$\mathbb{E}^3_+$ ($z \geq 0$)

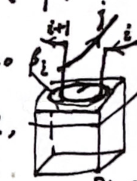


Knot grp $\pi_1(\mathbb{E}^3 - k)$ ($\cong \pi_1(S^3 - k)$)

$= \{x_1, \cdots, x_w \mid r_1, \cdots r_{w-1}\}$.

$\pi_1(\mathbb{E}^3_+ - k) = \underset{w}{*}\mathbb{Z}$
($= \pi_1(\mathbb{E}^3 - \hat{k})$)

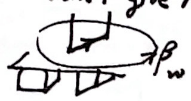
(Pf:  B_i, Van kapmann) $\cong$ cylinder $\mathbb{Z}$

(for $\mathbb{E}^3_- - k$, )
Van kapmann.
D_i (simply conn.)

(thus Ab(knot grp) = inf. cyclic grp $\mathbb{Z}$)

$r_i: x_j x_i = x_{i+1} x_j$ if (otherwise x_j^{-1} i.e. $x_i x_j = x_j x_{i+1}$)

(note: the last underpass didn't give r_w since $e = e$)



$x_i x_j x_{i+1}^{-1} x_j^{-1} = e \longleftarrow r_i$

(if no overpass above, underpass $x_i = x_{i+1}$)

ex. trefoil 

$G = \{a, b \mid aba = b\} \cong S_3$. sym. grp

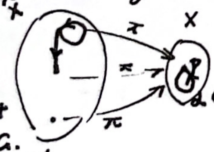
Seifert surface ori.

genus of knots $g(k)$
($g(k) = 0$ iff. no knot)

Knot sum 
$g(k + l) = g(k) + g(l)$
(thus we can't tie two knots into a string so that they cancel one another)

Lift $\pi \circ \tilde{f} = f$

(covering $\pi: \tilde{X} \to X$)



in same right coset of $\pi_*(H) \subset G$.

induced morph. $\pi_*: \pi_1(\tilde{X}, q) \to \pi_1(X, p)$ is inj. ($\pi(q) = p$)

$\forall x \in X$, $|\pi^{-1}(x)| = [\pi_1(X,p) : \pi_*(\pi_1(\tilde{X}, q))]$ $\left(= \dfrac{|\pi_1(X,p)|}{|\pi_*(\pi_1(\tilde{X}, q))|}\right.$ Lagrange$\left.\right)$

△ grps $\pi_*(\pi_1(\tilde{X}, \tilde{x}))$, $\tilde{x} \in \pi^{-1}(p)$, form a conjugacy class of subgrps of $\pi_1(X, p)$.

n-fold covering sp.

ex. $f(z) = z^n$ is n-fold on $\mathbb{C} - \{0\}$.
S^2 is 2-fold map sp. of P^2. ($[\mathbb{Z} : n\mathbb{Z}]$)

Map Lift Th: $f(r) = p$ lifts to $\tilde{f}(r) = q$. unique. iff. $f_*(\pi_1(Y, r)) \subset \pi_*(H)$.

$$\begin{tikzcd} & H = \pi_1(\tilde{X}, q) \arrow[d, "\pi_*"] \\ \pi_1(Y, r) \arrow[ur, "\tilde{f}_*"] \arrow[r, "f_*"] & \pi_1(X, p) = G \end{tikzcd}$$

$H = \pi_1(\tilde{X}, q)$.

equivalence:

$\exists$ mor $h: \tilde{X}_1 \to \tilde{X}_2$, $\pi_2 \circ h = \pi_1$.

($\pi_{2*}(H_2) = \pi_{1*}(H_1)$) ($\Leftrightarrow$ 确定X π_1内同一个子群共轭类)

(hier. $\tilde{X}_2$ — $\pi_{2*}(H_2)$, $\tilde{X}_1$ — $\pi_{1*}(H_1)$, X — G)

△ $\pi: \tilde{X} \to X$, if $\pi \circ h = \pi$ and $h(\tilde{x}_0) = \tilde{x}_0$ then $h = 1_{\tilde{X}}$ since lifting uniqueness.

Covering transformation mor. $\pi \circ h = \pi$
$\longrightarrow$ forms a grp K.

($H = \pi_1(\tilde{X})$)
if $\pi_*(H) \triangleleft G$, then $K \cong \dfrac{G}{\pi_*(H)}$, $X \cong \dfrac{\tilde{X}}{K}$.

$\alpha \sim \beta$: $\alpha\beta^{-1}$ null-mot.
{道路等价类}

Orbit sp. $\dfrac{\tilde{X}}{K}$

regular covering
↑
if $\pi(\tilde{x}_1) = \pi(\tilde{x}_2)$, $\exists h \in K$ s.t. $h(\tilde{x}_1) = \tilde{x}_2$.

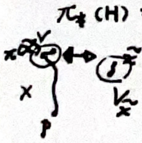
ex. 1. $\mathbb{R} \to S^1$. $K = \mathbb{Z}$.
2. $\mathbb{C} - \{0\} \to \mathbb{C} - \{0\}$. $K = \mathbb{Z}_n$.

universal covering sp. $\exists$. if conn...

$\pi_*(H) = \{e\}$, $K \cong \pi_1(X)$. $H \subset \pi_1(X)$, $\dfrac{\tilde{X}}{H}$ also a covering with $\pi_1(\dfrac{\tilde{X}}{H}) \cong H$.

($\pi_*(H)$ 表征$\tilde{X}$中环道在X里的表现, X中环道走到了$\tilde{X}$中的一类点, 由K将其粘合 (一个元素对一个))

(uniqueness since $\tilde{x}$, α, let $k_\alpha: q \mapsto \tilde{\alpha}(1)$,
$\langle \alpha\beta^{-1} \rangle \in \pi_*(H) \Leftrightarrow k_\alpha$ same.)



Alexander polynomial.

ex. trefoil
$t^2 - t + 1$.