Game Theory

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2024.4

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1 Shortcomings

This note is a homage to the classic theory of game and reframes it, because of some weakness in describing some situations. Here I show three cases and discuss possible improvement for further studies.

- **Imcomplete information duopoly.** Producer 1 knows his cost (either high or low) while doesn't know whether producer 2 knows this cost, and producer 2's cost is known by both.

  Actually there is no defect within Harsanyi's theory, but it's quite tricky for beginners. If we think in a way that producer 1 has two types - High or Low, and producer 2 also has two types - Know or Unknow, then under Harsanyi's hypothesis, the joint distribution is common knowledge. However if it is, then producer 2 has already known the cost(from conditional distribution at 'Know').

  The solution is that producer 2 has three types - 'Know High', 'Know Low', 'Unknow'. In other words, there are three information sets i.e. (High, Know High), (Low, Know Low), {(High, Unknow), (Low, Unknow)}.

- **Unrationality.** A classical dynamic game is two real estate developers deciding whether to build new houses. The later one won't build because he shouldn't hurt himself. Some situations in our life have told us, however, he may fight to both go up. You may say that's easy enough, since we can always substitute new utility or payoff for old one, as long as an utility function can exist to describe his preference. Like in this case, we can subtract another developer's payoff from his, but in some cases—some lexicographical preferences, for example—we can't. Or if payoff involves uncertainty, not everybody will use expectation directly. An usual measurement may be the value of expectation minus variance by a certain weight just as investment.

- **Uncertainty from nature.** Not all uncertainty comes from other's personal information. Players might not be able to tell two possible outputs, and they hold different beliefs of the probability distribution.

2 Social Mechanics

Let me talk about some broader views. What game theory would be depends on how we perceive game.

The definition of science has controversy. A general one is using scientific approach, e.g. experiment, induction and deduction to acquire knowledge, whereas most people still only call those theories with formal derivation (strictly, Turing computable or Newtonian reductionism) science. Now there is a syllogistic reasoning: 1.Economics is science. 2.Science has computable, mechanics-like theory. Therefore, 3.Economics should expand as 'social mechanics'.

This term refers to a deterministic world. Though it seems a bit biassed, we need acknowledge that modern economics and game theory have already been endeavoring to get close to that. People should not bury their head in the sand.

Below I list three theorems imitating for mechnics in physics.

Theorem 1. ***First Theorem in Social Mechanics.***

$$G \xrightarrow[S]{N} G'$$

*Our system consists two parts, i.e. nature N and human(society) S. G is a status(including people's memory - stored in their brain cells) of the system. Newtonian view will predict the future world by this present status, not the past. Thus we can get a **Transition Function** $T : G \times t \rightarrowtail G'$, with $T(G,t) = T(T(G,\tau), t-\tau), \forall t, \tau$.*

*For discrete time, we can equally define a **Reaction Function** T that $T(G) = G'$ at each step. And we also propose that if we can separate time into sufficiently many parts, at which either nature or human act on the system i.e. reaction function $T = S \cup N$ can be welded from S and N.*

Theorem 2. ***Second Theorem in Social Mechanics.***

S and N's reaction at any status depends on their beliefs. Note that 'belief' contains the cognition of G and behavioral belief of what system parts will do.

Nature is regarded as complete and precise cognition of status and 0-th belief.(Only know what he will do at this environment.)

Human's belief consists of three parts: status, behavior and criterion. These depict inference and recollection process. First we need a cognition(may be wrong or uncertain), then possible actions and reactions(belief of behavior) will lead the world's envolvement, at some point we would evaluate the status (e.g. expected utility).

Criteria include physical laws, how people deciding their reactions and how people form or revise their beliefs. Thus criteria induce the right(one believe it's most possible) status cognition and behavior—this condition for the belief is called consistency(see Sec.5).

The above paragraphs illustrate how elements in the system react. In short, it has no difference with Newton's Second Law, except abiding by unspecified criteria instead of $f = ma$.

Theorem 3. ***Third Theorem in Social Mechanics.***

Nature and human may act in nondeterministic ways. They can be described by probability.

3 Description of Game

Since Sec.1 mentions that we shall distinguish decision-making process and payoff, criterion will be used to determine the final evaluation of outcome, not the amount of payoff.

Definition 1. ***Game Set*** *is a set of several possible payoffs.* $G = \{u_1, \ldots, u_n\}$.

Remark 1. u_i, u_j *may give the same amout of payoff, but here we use different elements to denote for they are reached by different paths of actions.*

u can be seemed as an **assignment game** and is carried out by nature $\boldsymbol{n}$ to get payoffs(e.g. $\boldsymbol{v} \in \mathbb{R}^m$), but it is not always necessary.

If a game only has a set of payoffs, it can only express a single choice made by one player. Hence we need other structures to be added on this G.

Extension structure. Every game designate one player to choose among its direct subgames. Then the situation converts to another game, the subgame of previous games, and continue on. Obviously from the definition above, subgame is a subset of the original game, i.e. $G' \subset G$. The subgames of G' are also the subgame of G. All subgames of G form a set called subgame set $\tilde{G}$, which expresses the extension structure.

Definition 2. ***Subgame Set*** *is a family of subsets of game G, satisfying*

1. $G \in \tilde{G}, u \in \tilde{G}, \forall u \in G$
2. $\forall G_1, G_2 \in \tilde{G}, \exists G'_1, \ldots, G'_n \in \tilde{G} \,\text{s.t.}\, G_1 - G_2 = G'_1 \cup \cdots \cup G'_n$ *or* $\varnothing$
3. *If* $G_1, G_2 \in \tilde{G}, G_1 \cap G_2 \neq \varnothing$, *then either* $G_1 \subset G_2$ *or* $G_2 \subset G_1$.

Condition 2 means any difference set can be expressed by some subsets' union, so $\tilde{G}$ is a semi-field. If a subgame set of a game satisfies condition 3 then we say it's in **extensive form**.

Lemma 1. ***Extension Lemma.*** *Every discrete game has an extensive form.*

The concept of **discrete game** refers to those games with discrete domain of t in their transition functions. Otherwise it's a **continuous game**.

Main idea of the proof is that if $G_1 \cap G_2 \neq \varnothing, G_1 \not\subset G_2, G_2 \not\subset G_1$ then we add a copy of $G_1 \cup G_2$ into the game set and let them belong to G_1 and G_2 separatedly. Repeat this procedure from the top game until assignment, so the original set can be converted to its extension form.

As long as all subgames are determined, we can find all 'child nodes' and 'parent nodes' of any game, as well as the greatest common subgame and the least common surgame. We define an operator ∂ so as to find the first layer of subgames of a game.

Definition 3. $\partial G = \{G' | G' \in \tilde{G}_r, G' \subsetneq G, \nexists G'' \in \tilde{G}_r \,\text{s.t.}\, G' \subsetneq G'' \subsetneq G\}$

G's ***family*** $G^f = \{G' | G' \in \tilde{G}_r, G' \subset G$ *or* $G \subset G'\}$

G's ***reflection*** *or* ***siblings*** $\hat{G} = \{G' | \exists G'' \,\text{s.t.}\, G', G \in \partial G''\}$

G_r means the root or whole game our analysis is set in ($G \subset G_r$). $\partial u = \varnothing$.

Corollary 1. ∂G *is a partition of* G.

Partition means $\bigcup\limits_{G_i \in \partial G} G_i = G, G'_i \cap G'_j \neq \varnothing, \forall i, j$.

Convention 1. *We will always use extensive form in analyzing discrete games. Hence all* $\cup$ *later in this notes can be substitued by* $\sqcup$*(disjointed union) if not specified.*

Designation structure. At each game, there is a player choosing his reaction. However we need to assume this is really the case, so the first part of **regular condition**(see below) is proposed: *there is only one player facing a game.* He will know and must know it's his turn at this game(through his cognition). Thus there exists a designation function $\rho : \tilde{G} \to I$ to determine which player is transforming the status. $I = \{1, \dots, m, \boldsymbol{n}\}$ is the player (index) set, where $\boldsymbol{n}$ denotes nature's turn. (It's insignificant that ρ can also be defined only on $\tilde{G} \setminus G$.)

Information structure. Not all game proceed in total certainty. We need to acknowledge that there can be several status undistinguishable for a player (but can have a propensity of some special status) due to ability or observation constraints, etc. These things should be preappointed by a information structure.

Definition 4. ***Information Set*** *is a partition of subgame set* $\tilde{G}$. $\mathscr{T} = \{\tau_1, \dots, \tau_n\}$. ***(Information) Node*** $\tau = [G]$ *is the* ***equivalence class*** *of game* G.

We assume that distinguishability is an equivalence relation. And $[u] = u$ since nature is omniscient.

Let $\partial\tau = \bigcup_{G\in\tau} \partial G, \tilde{\tau} = \bigcup_{G\in\tau} \tilde{G}, \tau^f = \bigcup_{G\in\tau} G^f, \hat{\tau} = \bigcup_{G\in\tau} \hat{G}$.

Definition 5. ***Regular Condition*** *or* ***Complete Confronting Condition****:*

1. *Designation* ρ *is a function, e.g.* $\rho(G) = i$.
2. *If* $G_1, G_2 \in \tau$, *then* $\rho(G_1) = \rho(G_2) = \rho(\tau)$.
3. *If* $G_1, G_2 \in \tau$, *then cardinality* $|\partial G_1| = |\partial G_2| = \frac{|\partial\tau|}{|\tau|}$.

We call $N(\tau) = \frac{|\partial\tau|}{|\tau|}$ the **choice number** of node τ.

Note that since $\rho([G]) = \rho(G) = i$, $\mathscr{T}$ actually can be written as $\bigcup_{i\in I} \mathscr{T}_I$, i.e. $[\ ]$ can be specified as $[\ ]_i$.

Remark 2. *Extended form can be represented by a directed tree graph. Information set can be represented by a directed graph (with only one spanning tree).*

Definition 6. ***Strategy*** *is a reaction function.* $S : \tilde{G} \to \tilde{G}$.

We artificially set $S(u) = u$ for assignment games, thus can always get $u = S^\infty(G)$. Iterate S to define $\bar{S}$ or $S^\infty : \tilde{G} \to G$.

Notice that since games in a same τ can't be separated by player $\rho(\tau)$, he will have the same cognition and all other beliefs under this circumstance, and therefore his reaction will be the same on a τ. $\partial\tau$ is split into $\partial\tau_1, \dots, \partial\tau_n$, where $n = N(\tau)$ is the choice number. $S_{\tau i} : \tau \rightarrowtail \partial\tau_i$. Strategy subspace $\mathbb{S}_\tau$ at this node also has a cardinality of $N(\tau)$. Then the whole strategy space $\mathbb{S} = \bigoplus_{\tau\in\mathscr{T}} \mathbb{S}_\tau$ is the direct sum of each node's strategy space. (Strictly to say we need to define a zero element in $\mathbb{S}$ but it's arbitrary.)

Remark 3. $\tilde{G}$ *or* ∂ *is slightly weaker than* $\mathbb{S}$*, since that it doesn't define how subgames of* τ *being categorized by player's choices. We will still use* $\tilde{G}$ *if no ambiguity.*

For mixed strategy, we can write $S : \tilde{G} \to \mathscr{D}(\tilde{G})$ with $\operatorname{supp} S(G) \subset \partial G, \forall G \in \tilde{G}$. $\mathscr{D}(\tilde{G})$ is the distribution space on $\tilde{G}$ (as pdf P in probability space $(\tilde{G}, 2^{\tilde{G}}, P)$). $\bar{S} : \tilde{G} \to \mathscr{D}(G)$. And we may write mixed strategies in pro forma addition(weighted by probability): let $P(i) = S_\tau(G; G_i'), G_i' \subset G \in \tau, G_i' \in \partial\tau_i$, then $S_\tau = \sum_{i=1}^{N(\tau)} P(i) S_{\tau i}, \sum_i P(i) = 1$, which defines an addition operation in subspace $\mathbb{S}_\tau$.

Remark 4. *The concept of neighborhood can be generated from the definition of strategies' distance* $d(S_1, S_2) = \max_{G'\in\partial G, G\in\tilde{G}} |S_1(G; G') - S_2(G; G')|$.

No information structure when every subgame is their own info set. At this point we call $\mathscr{T}$ **trivial information** or **perfect information** as in traditional game theory.

A real game can be written as a tuple $(G, \tilde{G}, \rho, \mathscr{T})$. Actually $\mathscr{T}$ brings the information in G and $\tilde{G}$. Sometimes we say 'game G' if there is no ambiguity.

Ordered form can omit designation structure as well. Under this form, players will play the game in turn though there are some nodes that only have one choice. Especially, all nature $\boldsymbol{n}$'s rounds will be lifted and combined into the root round at the beginning of G_r.

Definition 7. *A game G_r is in **ordered form** if it satisfies*

1. $\forall G \in \tilde{G}_r, G_1, G_2 \in \partial G, \rho(G_1) = \rho(G_2) = i$. *Write as* $\rho(\partial G) = i$.
2. $\forall G \in \tilde{G}_r \setminus G_r, \{\rho(G), \rho(\partial G), \ldots, \rho(\partial^{m-1} G)\} = \{1, \ldots, m\}$.

Lemma 2. ***Ordering Lemma.***

Every extensive form with finite player set has an ordered form.

Proof. The process of conversion includes three stages: delete, insert and combine.

1. Delete all G with $\partial G = G$.
2. Insert new G(single in $\mathscr{T}$) from $\rho(G)$ to $\rho(\partial G)$.(Only $\rho(\partial G) = \rho(G)$ or $\rho(G) + 1$ needn't.)
3. Combine G and ∂G if $\rho(G) = \rho(\partial G)$. □

Lemma 3. ***Nature Lifting Lemma.***

A game with nature player can be converted into a game where nature is only at the root and the end, i.e. $\rho^{-1}(\boldsymbol{n}) = G_r \cup \{G_r\}$.

Proof of this lemma and a stricter definition of 'convert' can be found in Sec.7.

Remark 5. *Harsanyi told us unceratinty can use a player $\boldsymbol{n}$ named nature to implement. Then lift all nature's rounds onto the root game. This 'uncertainty transference' process is the so-called **Harsanyi transformation**.*

We denote ∂G_r as $\mathbb{G} = \{G_1, \ldots, G_n\}$, which is the set of all structure-different games(now no nature player) that at least one player has uncertainty about. A classical incomplete information game is $(\mathbb{G}, \tilde{\mathbb{G}}, \mathscr{T}_{\mathbb{G}})$, where $\tilde{\mathbb{G}} = \bigcup\limits_{G \in \mathbb{G}} \tilde{G}, \mathscr{T}_{\mathbb{G}} = \bigcup\limits_{G \in \mathbb{G}} \mathscr{T}_G$. Harsanyi transformation shows it's identical with adding a nature, i.e. $(\mathbb{G}, \tilde{\mathbb{G}}, \mathscr{T}_{\mathbb{G}}) \leftrightarrow (G_r, \tilde{G}_r, \mathscr{T}_r)$, where $\tilde{G}_r = \{G_r\} \cup \tilde{\mathbb{G}}, \mathscr{T}_r = \{\{G_r\}\} \cup \mathscr{T}_{\mathbb{G}}$. (We'll mainly use G_r below.)

If $\mathbb{G}$ is a singleton($|\mathbb{G}| = 1$), then in traditional game theory we say this game $G = \mathbb{G} = G_r$ is **complete information**.

If $\hat{G} = [G]$ (or $\hat{\tau} = \tau$), then the action of the player at parent game (or node) is not observed by the player at this game(i.e. determine their choices simultaneously), so the two's actions are **commutative** or G (or τ) is **(sectionally) static**.

Definition 8. ***Direct sum** of game G_1 and G_2 is constructed by*

$G = G_1 G_2 = \{u_1 u_2 | \forall u_1 \in G_1, u_2 \in G_2\}$

$\tilde{G} = (\tilde{G}_1 G_2) \cup (G_1 \tilde{G}_2)$

$\mathscr{T} = \mathscr{T}' \cup (G_1 \mathscr{T})$, *where* $\mathscr{T}' = \{\tau_1'(\tau_1) | \tau_1 \in \mathscr{T}\}, \tau_1'(\tau_1) = \{G_1' G_2 | G_1' \in \tau_1\}$.

(There is no difference in direct sum and product etc if the number of games is finite.) The product of assignments $u_1 u_2$ means the payoffs will be added up. If times constant e.g. kG, then it means all payoffs $u \in G$ will be enlarged by k times. As for repeated game, we can define the concept of splitting.

Definition 9. *If* $\exists G_1, \dots, G_n \in \tilde{G}$ *satisfy*

1. $\bigcup_{i=1}^{n} G_i = G$.
2. $\forall i, j \in \{1, \dots, n\}, G_i \cong G_j$.
3. $G_i, i \neq 1$ *are* G_1*'s cosets, i.e.* $\exists u_1 = 0, u_2, \dots, u_n \, \text{s.t.} \, G_i = u_i G_1$.

Then $G = G^* G_1$ *with* $G^* = \{u_1, \dots, u_n\}, \tilde{G^*} = \{G'^* | \exists G' \in \tilde{G} \, \text{s.t.} \bigcup_{j=1}^{k} G_{i_j} = G', G'^* = \{u_{i_1}, \dots, u_{i_k}\}\}, \mathscr{T}^* = \{\{G^*_{\tau 1}, \dots, G^*_{\tau k}\} | \tau = \{G_{\tau 1}, \dots, G_{\tau k}\} \in \mathscr{T}\}$.

(Refer to Sec.7 for the isomorphism in condition 2.)

4 Belief System

Belief is the modelizing of one's thinking or reasoning. It's impossible to slove games universally if we do not include depictions of epistemological things. However we can not expect an explanation for every subtle behavior of a person. Though most people will only reason at most one or two level at their daily life, we should formulate a roughly rational person(sometimes with some behavioral properties) making his decision through complete inference to work out our theory in an easy and realizable way.

0-th belief refers to 'I think ...'. 1-st belief refers to '(I think) he think ...'. 2-nd belief refers to '(I think) he think others think ...', and so on. B_i^0 denotes 0-th belief of player i. B_i^n is the n-th belilef of player i, but it needs n parameters to indicate the inference path(then we can omit n), e.g. $B_i^2(j, i)$—i thinks what j will think about himself.

Definition 10. ***Belief*** *of player* i *is a function* $B_i : F(I) \to \mathbb{B}$*, satisfying* $B_i(i * p) = B_i(p), \forall p \in F(I)$. ***Belief System*** *on* $I = \{1, \dots, m\}$ *is a tuple of belief functions* $(B_1, \dots, B_m)$.

$F(I)$ is the free group generated by player set I. $\mathbb{B}$ is the abstract space of possibly believed things(e.g. strategy). $B_i(\varnothing) = B_i^0 = B_i^1(i)$.

We focus on infinite-class beliefs. There can be three cases: fixed, periodic and chaotic. In some games like matching pennies there exist periodic infinite beliefs, but we won't discuss the latter two cases.

Definition 11. ***Common Belief*** *is a belief satisfying* $\forall j, p, B_i(p * j) = B_i(j)$.

Hence $Im(B_i) = \{B_i(1), \dots, B_i(m)\}$. If $B_i(j)$ is really what j thinks (maybe through some mechanisms like pre-communication), then $(B_1^0, \dots, B_m^0)$ is commom knowledge, or CK.

Definition 12. ***Common Knowledge*** *is a belief system that has* $B_i = B_j = B, \forall i, j$.

In other words, everyone knows (everyone knows ..., and so on) i thinks of $B(i) = B_i^0$. B may even be a consant map, i.e. $B(p) \equiv b$, then we say something b(e.g. physical laws for game players) is CK. Obviously, this CK system $(B_1, \dots, B_m)$ itself is CK.(Those belief functions B_i can also become the believed stuff.)

Remark 6. *CK must be CB. Just repeat* $p = k * p', k \in I, B_i(p * j) = B_k(p' * j) = B_i(p' * j)$ *to get* $B_i(j)$*. The only difference is using* $B_i(j)$ *or* $B_j(B(j))$*. However, sometimes we don't make it so clear that it's CB or CK since our analysis can either set in one player's mind or in the reality.*

5 Principles

There are four principles proposed when we are formalizing the theory.

Principle 1. ***Cognition Principle.***

Root game $(G_r, \tilde{G}_r, \mathscr{T}_r)$ is CK.

Now let's put beliefs in the context of social mechanics. For nature, $B_{\boldsymbol{n}} = B^0_{\boldsymbol{n}} = (C_{\boldsymbol{n}}, S_{\boldsymbol{n}})$($C_{\boldsymbol{n}}$ just dictates $S_{\boldsymbol{n}}$). For human, related beliefs contains three aspects(see Sec.2): criteria c, reactions s and status h.

We will formulate one's criterion as two steps: cognition and valuation. At the first step, a cognition is derived from the (believed) system's reaction functions starting from the genesis(root) status with the criterion. At the second step, the criterion tells what's the best reactions given present status and others' future reaction functions.

Definition 13. ***Cognitive Criterion*** *at node τ is a function $C_H : \mathbb{S}_r \to 2^{\mathscr{D}(\tau)}$, where $\mathbb{S}_r$ is the strategy space from the root game $\tau_r = \{G_r\}$ (or $\subset \mathbb{G}$).*

Strategic Criterion *at node τ is a function $C_S : \mathscr{D}(\tau) \times \mathbb{S} \to 2^{\mathbb{S}}$, where $\mathbb{S}$ is the strategy space from node τ (till assignment).*

(Strictly to say $\mathbb{S}_r$ in C_H's domain should be $\mathbb{S}_r/\mathbb{S}$, and $\mathbb{S}$ in C_S's domain should be $\mathbb{S}_{-i} = \mathbb{S}/\mathbb{S}_i$, where $i = \rho(\tau)$.)

Unlike the physical world, however, our criteria are often not complete enough, which means that there exists a indifference set of choices. Hence we use $2^{\mathscr{D}(\tau)}$ and $2^{\mathbb{S}}(2^{\mathbb{S}_i})$ as the codomains.

Remark 7. *Those domains and codomains can be slightly expanded with some freedom. For example, if $C^*_S : \mathscr{D}(\tau) \times \mathbb{S}_{-i} \to 2^{\mathbb{S}_i}$, we can define $C_S(f, S) = C^*_S(f, S_{-i}) \oplus S_{-i}$ to get C_S.*

We want the synthetic criterion to look at all possibly ocurred strategy choices. Noting that space $\mathbb{S}$ is a part of space $\mathbb{S}_r$, let p be the projection onto $\mathbb{S}$.

Definition 14. ***(Synthetic) Crietrion*** *is a compsite function of C_H and C_S. $C : \mathbb{S}_r \to 2^{\mathbb{S}}, C(S) = \bigcup_{f \in C_H(S)} C_S(f, p(S))$.*

Remark 8. *As for neoclassical economics, people make decisions by utility(if have a well-defined preference). A criterion function can also evaluates the final assignment from a game status through a specified reaction function, i.e. $C : \mathbb{S}_r \to \mathbb{U}$ ($\mathbb{U}$ is the utility space, e.g. $\mathbb{R}^m$). Then we using these values to determine the feasible region.*

At root node a player has a belief of $B_i = (c_i, s_i)$, but most of games have extension structure—we need to extend this initial belief throughout the whole game. (Notice that these lowercase letters are belief functions, and s_i is not left out due to the incompleteness of c_i.)

Definition 15. ***Sub-Criterion Function*** *$C : \mathscr{T} \times \mathbb{S} \to 2^{\mathbb{S}}$.* ***Sub-Strategy Function*** *$S : \mathscr{T} \times \tilde{G} \to \mathscr{D}(\tilde{G})$, $\operatorname{supp} S(G, G') \subset \partial G', \forall G' \subset G \in \tilde{G}_r$.*

(Strictly to say S's domain should be $\bigcup_{G \in \tilde{G}_r} (G, \tilde{G})$.)

We use $\mathscr{T}$ not $\tilde{G}$ to express that the player couldn't distinguish those statuses in one node.

Remark 9. *As mentioned earlier, we can also specify what cognition a player has. Sub-cognition function $H : \mathscr{T} \times \mathbb{S}_r \to \mathscr{D}(\tilde{G}_r), \operatorname{supp} H(\tau, S) \subset \tau, \forall \tau, S$. And from Principle 1(&3) we trivially know that $H(\tau_r)$ is CK. The consistency condition in Sec.6 can also be written in two parts (for C_H and C_S).*

Strategy function $S(G_r)$ is defined on the subgame set, and in substrategy we automatically keep its action on subgames.

Principle 2. ***Extension Principle.***

$\forall \tau', \tau \in \mathscr{T}, G \in \tau, \tau', S(\tau, G) = S(\tau', G)$.

$\forall \tau', \tau \in \mathscr{T}, \tau' \cap \tilde{\tau} \neq \varnothing, \rho(\tau) = \rho(\tau'), \forall S \in \mathbb{S}, C(\tau', C(\tau, S)) \subset C(\tau, S)$.

Due to this principle, substrategy $S(\tau) \in \mathbb{S}(\tau)$ has no difference from strategy $S \in \mathbb{S}$, so we won't separate these two S if no ambiguity.

Let $C(\mathscr{T}, S) = \bigcap_{\tau \in \mathscr{T}} C(\tau, S), C(\tau, \mathbb{S}) = \bigcap_{S \in \mathbb{S}} C(\tau, S)$. If a criterion satisfies the above condition then we call it **sequential criterion**(e.g. sequential rationality). Obviously, subcriterion that traverses the game itself can be packed into a criterion and it's sequential. And **strong sequential criterion** refers to $C(\tau', C(\tau, S)) = C(\tau, S)$.

Principle 3. ***Harsanyi Principle.***

Nature's belief $B_{\boldsymbol{n}}$ is CK. (The probability space $\mathbb{G}$ is CK.)

(After some transformations. Precise to say, $\mathbb{G}$ don't need to be a probability space since we only need distributions on each equivalent class, but in most cases we can derive a joint distribution.)

Here CK refers to the system $I = \{1, \dots, m\}$ without nature. This is probably the most tricky thing in game theory and has gone through extensive discussions by previous scholars.

Principle 4. ***Rationality Principle.***

$(r) \subset C$.

Perfect Recall *or* ***Complete Recalling Condition*** *for* $\mathscr{T}$: $\forall [G_1'] = [G_2'], \rho(G_1') = \rho(G_2') = i$, if $G_1' \subset G_1, B_2' \subset G_2, \rho(G_1) = \rho(G_2) = i$, then $[G_1] = [G_2], G_1 \cap G_2 = \varnothing$.

($G_1' \neq G_2'$, but G_1, G_2 can equal G_1', G_2'.)

(r) is the notation of rationality or Nash criterion(see Sec.6). The inclusion relation between criteria is defined by their completeness.

Definition 16. *If $\forall \tau, S, C_1(\tau, S) \subset C_2(\tau, S)$, then $C_1 \subset C_2$ or C_1 is a* ***refined criterion*** *relative to C_2.*

Rationality hypothesis is just what economists love to do, but there can be a more reasonable explanation that in most common cases we can rewrite payoffs to use rationality. Perfect recall is seemed as a rational behavior and it brings favorable properties, e.g. if $\tau' \cap \tilde{\tau} \neq \varnothing, \rho(\tau) = \rho(\tau')$ then $\tau' \subset \tilde{\tau}$.

Theorem 4. ***Kuhn's Theorem.***

Every mixed strategy $\sum_i P(i) S_i$ is equivalent to a behavioral(probabilistic) strategy under the assumption of perfect recall, i.e. $\sum_i P(i) S_i = \bigoplus_\tau S_\tau, S_\tau = \sum_i P(\tau, i) S_{i,\tau}, \sum_i P(i) = \sum_i P(\tau, i) = 1$.

$$P(\tau, i) = \begin{cases} \dfrac{P(i) P_{S_i}(\tau)}{\sum_i P(i) P_{S_i}(\tau)} & \sum_i P(i) P_{S_i}(\tau) \neq 0 \\ P(i) & \sum_i P(i) P_{S_i}(\tau) = 0 \end{cases}$$

where reaching probability function $P_S(G) = \prod_{G \subset G' \in \partial G''} S(G''; G'), P_S(\tau) = \sum_{G \in \tau} P_S(G)$ is induced by S.

6 Equilibrium

People organize their beliefs logically and reasonably.

Definition 17. ***Consistency Condition*** *of a player's belief $B_i = (c_i, s_i)$:*

$c_i(p; s_i(p)) \ni s_i(p), \forall p \in F(I)$.

('Cognition-reduced consistency'.) Here s is the belief of whole strategy of all players, so it satisfies $p_j(s_i(p)) = p_j(s_i(p * j))$, where $p_j : \mathbb{S} \to \mathbb{S}_j$ is the projection.

Since belief at τ_r can extend to its subnodes, we shall let it equilibrate on all these nodes when reasoning.

Definition 18. ***Equilirium*** *$\pi = (C, S)$ is a consistent commom belief throughout the game. Under CK($B_i = B_j = B \equiv (C, S)$), root form and Principle 2, the condition equals $\forall \tau \in \mathscr{T}_r, S \in C(\tau, S)$.*

Remark 10. *As for subgames, the status contains not only G but also the initial cognition $H([G])$. The subgame should be induced by the original game and maybe go through a tedious procedure of separating and lifting to get $\mathbb{G}$. If we lift it to a new sub-root game G_r then equilirium (C, S, H) on G is equivalent to (C, S_r) on G_r.*

Now let me specify some common criteria:

- (r) **Nash(Rational) criterion**. If $\tau = \{G\}$, then $C(\tau, S) = \{S^* \in S_{-\tau} \oplus \mathbb{S}(\tau) | \bar{S}^*(G)_i \geq \bar{S}'(G)_i, \rho(\tau) = i, \forall S^*_{-i} = S'_{-i} = S_{-i}\}$. ($\bar{S}(G)_i$ denotes the i-th payoff u_i of $u = \bar{S}(G)$. $S_{-\tau}$ denotes the projection of S onto all $\tau' \in \mathscr{T}, \tau' \cap \tilde{\tau} \neq \varnothing$.) The inequality can be restricted to only neighborhood $S' \in S^{*\sim}$.

- (re) is not frequently used. It only states that we can calculate and compare the expectation when facing with mixed strategy and a probabililty distribution of payoffs.

- (rB) **Nash-Bayesian criterion**. $C(\tau, S) = \{S^* \in S_{-\tau} \oplus \mathbb{S}(\tau) | \sum_{G \in \tau} H(\tau, G)\bar{S}^*(G)_i \geq \sum_{G \in \tau} H(\tau, G)\bar{S}'(G)_i, \rho(\tau) = i, \forall S^*_{-i} = S'_{-i} = S_{-i}\}, H(\tau, G) = \frac{P_S(G)}{P_S(\tau)} = \frac{P_S(G)}{\sum_{G \in \tau} P_S(G)}$. (Or use integral instead of summation.)

Actually infinite-class belief of strategy can only be consistent on IE(iterated elimination of strictly dominated strategies) set if rational(Principle 4), which induces **Nash equilibrium**.

Let **equilirium set** $\Pi(G)$ consists of all possible $\pi = (C, S)$ on G. Sometimes $\Pi_C(G)$ or $\Pi(G)$ denotes the set of all equilibrated S under a known C.

Lemma 4. ***Refinement.*** *If $C_1 \subset C_2$, then $\Pi_2 \subset \Pi_1$.*

And here is the rewriting of some famous conclusions in game theory:

- **Folk Theorem.** Let $\bar{\pi}(G) = \bar{S}(G)$ and $\bar{\Pi}(G)$ be the set of all possible equilibrated outcomes. $C(G) = \bar{\Pi_C}(G) \subset \bar{\Pi}(G)$ denotes the set of all equilibrated outcomes from criterion C. In finitely repeated games (with discounting rate δ), $r_\delta(G^n) = r(G * \delta G * \cdots * \delta^{n-1} G) = \begin{cases} \dfrac{1-\delta^n}{1-\delta} r(G) & \delta \neq 1 \\ nr(G) & \delta = 1 \end{cases}$.

  For infinite repeated games G^∞, folk theorem tells us that all feasible payoffs above single game G's Nash equilibria(actually minimax payoff) can be achieved as equilibrium with a large enough δ.

  If $u \geq r(G), u \in \bar{\mathbb{S}}(G)$, then $\exists \delta_0 < 1, \forall \delta_0 < \delta < 1, u \in (1-\delta) r_\delta(G^\infty)$.

- **Purification Theorem.** Define $\mathbb{G} \to G_0$ as $\forall G \in \mathbb{G}$, G and G_0 gradually converge(same payoffs).

  $\Pi_r(G_0) = \lim_{\mathbb{G} \to G_0} \sum_{G \in \mathbb{G}} P(G) \Pi_r(\mathbb{G}; G)$.

  (Note that $P(G)$ is CK and $\Pi_r(\mathbb{G}; G)$ is the equilibrated strategy function on game $G \in \mathbb{G}$.)

  Proof can be obtained through $\mathscr{T} \to \mathscr{T}_0, S \subset C(\mathscr{T}, S)$, merging $\tau_1, \ldots, \tau_n$ into τ and $S_0(\tau) = \sum_i \frac{P_S(\tau_i)}{\sum_i P_S(\tau_i)} S(\tau_i)$, then prove $S_0 \subset C(\mathscr{T}_0, S_0)$.

- **Trembling Hand Perfect Equilirium.** Theorists have long been striving to find better refinements. One of the achievements among these is the so-called 'trembling hand', similar to a kind of strengthened continuity.

  This equilibrium is derived from trembling-hand criterion or **stable criterion**:

  For $S_0 \in C(\tau, S_0), \exists$ a neighborhood $S_0^\sim, \text{s.t.} \forall \tau, S \in S_0^\sim, S_0 \in C(\tau, S)$.

It is important to point out that refining equilibria with some properties can also be achieved by a stronger criterion satifying the corrensponding properties. Let $P_C(S)$ be the predicate of $S \in C(\mathscr{T}, S)$ and $\Pi_C = \{S \in \mathbb{S} | P_C(S)\}$.

Lemma 5. $\forall C, P, \exists C'$ s. t. $P_C(S) \wedge P(S) = P_{C'}(S)$.

We can firstly do the original C and then move out some S that dissatisfies the additional refining prediacte P in C_S's result. If $P(S)$ contains both S and C, then we may restrict the criterion by $P(C) = P(S, C)$, like stable criterion.

Now we will give the (sufficient) existence condition of a equilibrium.

Theorem 5. ***Equlibrium Existence theorem.***

Compact and convex $\mathbb{S}$, closed, convex and sequential criterion C has $\Pi_C \neq \varnothing$.

Note that $C(\tau, S)$ is a set-value function of S and closure means its graph is closed, i.e. $\{S_k\} \to S_0, \{S'_k\} \to S'_0$, if $S'_k \in C(\tau, S_k)$ then $S'_0 \in C(\tau, S_0)$. Convex criterion means $\forall \tau, S, C(\tau, S)$ is convex. The condition of closed criterion can be replaced by upper semi-continuity and $\forall \tau, S, C(\tau, S)$ is closed. (rB) and other normal criteria satisfy all these conditions.

Proof. We'll use Kakutani fixed point theorem. While $C(\tau, S)$ is nonempty, sequentiality preserves this property in $C(\mathscr{T}, S) = \bigcap_{\tau \in \mathscr{T}} C(\tau, S) = C(\{\tau_1, \ldots, \tau_n\}, S) \neq \varnothing$, where $\tau_i \cap \tau_j^f = \varnothing, \forall \rho(\tau_i) = \rho(\tau_j)$.

□

7 Map of Game

Let us look at two games G_1 and G_2. Here $\mathscr{T}_1, \mathscr{T}_2$ denotes all nodes $\tau \neq u$ and $\bar{\mathscr{T}}_1 = \mathscr{T}_1 \cup G_1, \bar{\mathscr{T}}_2 = \mathscr{T}_2 \cup G_2$.

Definition 19. ***Weak Strategic Homomorphism*** *is a map $\phi : \mathbb{S} \to \mathbb{S}'$ satisfying $\phi(\alpha S_1 + (1-\alpha) S_2) = \alpha\phi(S_1) + (1-\alpha)\phi(S_2), \forall S_1, S_2 \in \mathbb{S}$.*

Notice that $\mathbb{S} = \bigoplus_{\tau \in \mathscr{T}} \mathbb{S}_\tau$. There may be some cases when a subspace of $\mathbb{S}_1$ is mapped into a subspace of $\mathbb{S}_2$, which naturally induces a correnspondence between two information sets. We can decompose the space of $\mathbb{S}_1$ and $\mathbb{S}_2$ to build a map between those subspaces as finely as possible.

Lemma 6. ***Decomposition.*** *$\exists$ a partition of $\mathscr{T}_1$ and $\mathscr{T}_2$, s. t. $\forall [\tau_1], \exists [\tau_2], \phi(\bigoplus_{\tau \in [\tau_1]} S_\tau) \subset \bigoplus_{\tau \in [\tau_2]} S_\tau$, and this partition is finest for $\mathscr{T}_2$, i.e. $\forall [\tau_2] \subset \mathscr{T}_2, \nexists [\tau_2]' \subsetneq [\tau_2]$, s. t. $\exists \tau_1 \in \mathscr{T}_1, \phi(S_{\tau_1}) \subset \bigoplus_{\tau_2 \in [\tau_2]'} S_{\tau_2}$.*

(The worst situation is they are not divided at all.) Let $\phi^\tau : [\tau_1] \rightarrowtail [\tau_2]$ be the map of these decomposed node groups.

In order to see what relation exists between two corresponding groups of nodes, we expand the definition of choice number to a set of nodes in a revursive way.

Theorem 6. ***Choice Number Formula.***

$N(\tau_1, \ldots, \tau_n) = N(\tau_n) N(\tau_{-n}) - (N(\tau_n) - 1)(N(\tau_n^f \cap \tau_{-n}) - 1)$. ($\tau_{-n} = \{\tau_1, \ldots, \tau_{n-1}\}$.)
If $\phi : S_{\tau_1^1} \oplus \cdots \oplus S_{\tau_n^1} \to S_{\tau_1^2} \oplus \cdots \oplus S_{\tau_m^2}$ is a biject, then $N(\tau_1^1, \ldots, \tau_n^1) = N(\tau_1^2, \ldots, \tau_m^2)$.

It's easy to see $N(\mathscr{T}) = |\mathbb{S}|$.

Especially, if $\forall [\tau_1], \phi^\tau([\tau_1])$ is a singleton, i.e.$\forall \mathbb{S}_{\tau_1} \in \mathbb{S}_1, \exists \mathbb{S}_{\tau_2} \in \mathbb{S}_2, \phi(\mathbb{S}_{\tau_1}) \subset \mathbb{S}_{\tau_2}$, then it induces a function $\phi^* : \mathscr{T}_1 \to \mathscr{T}_2$, $\phi^*(\tau_1) = \tau_2$. And we expand it to $\bar{\mathscr{T}}_1$, defining that $\phi^*(\bar{S_1}(G_1)) = \phi(\bar{S_1})(G_2)$(it's not unique for each u_1).

Definition 20. ***Structural Homomorphism*** *is a map $\phi^* : \bar{\mathscr{T}} \to \bar{\mathscr{T}}'$ satisfying $\phi^* \tau^f \subset (\phi^* \tau)^f, \forall \tau \in \bar{\mathscr{T}}$, where $\tau^f = \{\tau' \in \bar{\mathscr{T}} | \tau \cap \tilde{\tau'} \neq \varnothing\}$.*

Thus in this case we have $N(\tau) = N(\phi^{*-1}(\tau))$. If two games are structural homomorphic mutually, then we say they are (structrally) **isomorphic** $G_1 \cong G_2$. $|\mathscr{T}_1| = |\mathscr{T}_2|, |G_1| = |G_2|, \phi^*\tau^f = (\phi^*\tau)^f$.

Corollary 2. *Commuting two players' actions at a static game(node) is isomorphic.*

Definition 21. ***(Strong) Strategic Homomorphism*** *is a weak strategic homomorphism satisfying* $\phi(C_1([\tau_1], S_1)) \subset C_2(\phi^\tau([\tau_1]), \phi(S_1)), \forall \tau_1, S_1$.

(ϕ^τ is induced by ϕ.)

If two games' strategy space are mutually strategic homomorphic, then we way they are **isostrategic** $G_1 \sim G_2$. $|\mathbb{S}_1| = |\mathbb{S}_2|, C_2(\phi^\tau([\tau_1]), \phi(S_1)) = \phi(C_1([\tau_1], S_1))$.

Corollary 3. *For holding a same criterion that only considers the final payoff profile—not the location in game, for example—two games are isostrategic as long as the corresponding strategies give out the same amount of payoff.*

If $\forall \tau, S, C_1(\tau, S) = C_2(\tau, S) = \{S' | P(\tau; \bar{S}', \bar{S})\}, \bar{S}(G_1) = \phi(\bar{S})(G_2)$*, then* $G_1 \sim G_2$.

Obviously, if two games are isomorphic and $\phi^*(u) = u$, they must be isostrategic, but not vice versa.

If ϕ and the induced ϕ^* are both surjective homomorphisms, then we say that G_1 is the **refinement** or **lifting** of G_2 and G_2 is the **folding** of G_1.

Theorem 7. ***Refinement Theorem.***

If G_1 *is a refinement of* G_2*, then* $\phi(\Pi(G_1)) \subset \Pi(G_2)$.

Proof. Since surjection, $\mathscr{T}_2 = \bigcup_{\tau_1 \in \mathscr{T}_1} \phi^\tau([\tau_1])$. Let $\pi_1 \subset C_1(\mathscr{T}_1, \pi_1), \phi(\pi_1) \subset \phi(C_1(\mathscr{T}_1, \pi_1)) \subset C_2(\mathscr{T}_2, \phi(\pi_1))$, thus $\phi(\pi_1) = \pi_2$ is a equilibrium.

□

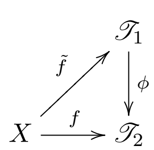
Hence isostrategic games have one-to-one correspondence between their equilibria $\phi(\Pi_1) = \Pi_2$.

Remark 11. *As for sequential criteria, if a* (C, S) *equilibrates on the highest node of every player, then it will also be equilibrated on all subnodes.*

Suppose $\mathscr{T}_N = \{\tau_1, \ldots, \tau_{|I|}\}, \phi^* : \mathscr{T} \to \mathscr{T}_N, \phi^*(\tau) = \tau_i, \rho(\tau) = i$ *and nodes obey choice number formula(existing a strategic bijection* ϕ*). Then let* $C = C_N$ *and* $\bar{S}(G) = \phi(\bar{S})(G_N)$.

This ***normal folding*** *or normal form in traditional game thoery is isostrategic (but not isomorphic) to the original extensive-form game if the criterion is sequential, and therefore the existence of Nash equilibrium ensures the existence of an equilibrium in* G.

Theorem 8. *A hom. map* f *can be lifted to a refined hom. map* $\tilde{f}$ *iff.* $f\Pi_X \subset \phi\Pi_1$.



Now let's prove the lemma left in Sec.3.

Proof. We'll show that $G \sim G'$.

□